Modal Logics from Coalgebras over Topological Spaces

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Abstract

This paper takes coalgebras for a functor as the principal models of modal logic. Varying the functor one obtains different categories of models. Using topological dualities and building on ideas of Johnstone and Abramsky, we show that a number of different modal logics can be derived in uniform way from their corresponding category of coalgebras.

Extended Abstract The historical development of modal logic proceeded from axiomatisations via algebraic semantics to Kripke semantics (frames and later (descriptive) general frames). This note shows that from a coalgebraic perspective, the logical order of the subject is the converse of the historical order.

Assume a concrete duality between categories $\mathcal{C} \simeq \mathcal{D}^{\mathrm{op}}$ and a functor $T : \mathcal{C} \to \mathcal{C}$. The category $\mathsf{Coalg}(T)$ of T-coalgebras is the category of the semantic structures we are interested in. The corresponding algebraic semantics is obtained by dualising T-coalgebras to T^{op}-algebras. Writing $K : \mathcal{C} \to \mathcal{D}$, $H : \mathcal{D} \to \mathcal{C}$ for the contravariant functors witnessing the dual equivalence $\mathcal{C} \simeq \mathcal{D}^{\mathrm{op}}$, we can define the dual of T on $\mathcal{D}$ as KTH. The very definition of coalgebras as the dual of algebras implies the duality $\mathsf{Alg}(KTH) \cong \mathsf{Coalg}(T)^{\mathrm{op}}$.

To obtain from this duality the language and axiomatisation of a modal logic, we describe the functor KTH by means of generators and relations. More precisely, we assume that $\mathcal{D}$ is a category of algebras and that free algebras exist. We then find $\hat{T} : \mathcal{D} \to \mathcal{D}$ such that $\hat{T} \cong KTH$ and, moreover, $\hat{T}(A)$ is a free algebra with respect to a set of generators G and equations E (see Table 1 for examples).

The generators give rise to a functor $L : \mathsf{Set} \to \mathsf{Set}$ which, intuitively, takes a set of formulas of rank (or depth) α and builds formulas of rank $\alpha + 1$. More precisely, L assigns to a set A the carrier of the free $\mathcal{D}$-object over the generators G. We can now generate the formulas of a modal logic by the so-called initial algebra sequence of L: Starting with the carrier $|D_0|$ of the initial object D_0 in $\mathcal{D}$, we consider the sequence $\langle L^\alpha(|D_0|)\rangle_\alpha$ for ordinals α and call elements of $L^\alpha(|D_0|)$ formulas of rank n.

Similarly, we consider the initial algebra sequence of $\hat{T}$ and the final coalgebra sequence of T. This gives the following picture where C_1 is the final object in $\mathcal{C}$, (X, ξ) is a T-coalgebra, and $\mathcal{P}$ denotes powerset.

$$X \xrightarrow{\xi_\alpha} T^\alpha(C_1) \qquad \mathcal{P}(T^\alpha(C_1)) \xleftarrow{[\![-]\!]} \hat{T}^\alpha(D_0) \xleftarrow{[-]_\equiv} L^\alpha(|D_0|)$$

ξ_α is the canonical projection assigning to each state in X its α-step behaviour, $[\![-]\!]$ is the function given by the duality $\mathcal{C} \simeq \mathcal{D}^{\mathrm{op}}$ which assigns to an element of the algebra $\hat{T}^\alpha(D_0)$ a subset of $T^\alpha(C_1)$.[1]

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[1] Here we assume that, as in all of our examples, the duality $\mathcal{C} \simeq \mathcal{D}^{\mathrm{op}}$ is induced by a schizophrenic object of cardinality 2.

We can thus define the semantics of the modal logic generated by L as follows. For a formula φ of rank α and a state $x \in X$ in a T-coalgebra (X, ξ) we put

$$x \models \varphi \;\Leftrightarrow\; \xi_\alpha(x) \in [\![[\varphi]_\equiv]\!]$$

Soundness and completeness are straight forward consequences of the duality.

Building on works of [Joh85] we apply the above framework to the category of T-coalgebras for T the Vietoris functor on topological spaces, using a duality result for $\mathcal{T}_0$-topological spaces [BJK95, BK99]. Table 1 shows some restrictions of this duality giving rise to well known soundness a completenss result of various modal logics.

Related Work [Joh82, Joh85] gives algebraic descriptions of the Vietoris space. The equations in the rightmost column of Table 1, for example, appear in [Joh85], 1.8.

[Abr91b, Abr91a] uses Stone duality to show that similar such equations axiomatise a sound and complete modal logic for certain kinds of transition systems. See also [Vic88]. Compared to their work, which is very much the foundation of ours, we use topological spaces like Stone spaces/spectral spaces instead of domains and coalgebras instead of solutions of recursive domain equations.[2]

The move to coalgebras is motivated by recent results showing that coalgebras for the Vietoris functor on Stone spaces are descriptive general frames [KKV03] and coalgebras for the Vietoris functor on Priestley spaces are $\mathbf{K}^+$-spaces [Pal03].

Future Work Two lines of research suggests themselves, namely to extend the present work to different base categories and to different functors. Concerning different base categories, we want to include in our uniform treatment of modal logics also intuitionistic modal logics. Concerning different functors, the first example that comes to mind is the (non-finitary) powerspace. In the case of the duality $\mathsf{Set} \simeq \mathsf{CABA}^{\mathrm{op}}$ this yields as coalgebras the Kripke frames, as modal algebras the complete atomic modal algebras of [Tho75] and as modal logic infinitary propositional modal logic. More generally, ongoing work will show how to extend our approach to functors T in general, using predicate liftings to generate modal operators [Pat01].

References

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[2] But recall that the domains of [Abr91b, Abr91a, Vic88] are particular spectral spaces. Moreover, the solution of a recursive domain equation $X \cong TX$ is a final coalgebra.

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Table of Examples We write Set, PoSet, Pri and Stone for the categories of sets, of partially ordered sets (isomorphic to the full subcategories of topological space with discrete and Alexandrov topology, respectively), of Priestley spaces (isomorphic to the category of spectral spaces) and of Stone spaces, respectively. Furthermore, we write CABA, CDL, BA, and DL for the categories of complete atomic boolean algebras, complete distributive lattices, Boolean algebras, and bounded distributive lattices, respectively.

For the functor $T : \mathcal{C} \to \mathcal{C}$ we are considering in this table the Vietoris space [Joh85]. For example, it correspond in Set to the finitary powerset. In the case of Stone spaces, the derived modal logic is the classical modal logic. In the case of Priestley spaces, the derived modal logic is positive modal logic [Dun95]. $\mathbf{K}^+$-spaces are due to [CJ99].

$\mathcal{C}$	PoSet	Set	Stone	Pri
Coalg(T)		image-finite frames	descriptive general frames	$\mathbf{K}^+$-spaces
$\mathcal{D}$	CDL	CABA	Stone	DL
Generators	$\Box a,\ a \in A$ $\Diamond a,\ a \in A$	$\Box a,\ a \in A$	$\Box a,\ a \in A$	$\Box a,\ a \in A$ $\Diamond a,\ a \in A$
Equations	$\Box \bigwedge_{i\in I} a_i = \bigwedge_{i\in I} \Box a_i$ $\Box \bigvee_{i\in I} a_i = \bigvee_{J\subseteq_{fin} I} \Box \bigvee_{j\in J} a_j$ $\Diamond \bigvee_{i\in I} a_i = \bigvee_{i\in I} \Diamond a_i$ $\Diamond \bigwedge_{i\in I} a_i = \bigwedge_{J\subseteq_{fin} I} \Diamond \bigwedge_{j\in J} a_j$ $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$ $\Box(a \vee b) \leq \Box a \vee \Diamond b$	$\Box \bigwedge_{i\in I} a_i = \bigwedge_{i\in I} \Box a_i$ $\Box \bigvee_{i\in I} a_i = \bigvee_{J\subseteq_{fin} I} \Box \bigvee_{j\in J} a_j$	$\Box(a \wedge b) = \Box a \wedge \Box b$ $\Box\top = \top$	$\Box(a \wedge b) = \Box a \wedge \Box b$ $\Box\top = \top$ $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$ $\Diamond\bot = \bot$ $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$ $\Box(a \vee b) \leq \Box a \vee \Diamond b$

Table 1: Table of Examples