Infinitary Domain Logic for Finitary Transition Systems

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Abstract. The Lindenbaum algebra generated by the Abramsky finitary logic is a distributive lattice dual to an SFP-domain obtained as a solution of a recursive domain equation. We extend Abramsky's result by proving that the Lindenbaum algebra generated by the infinitary logic is a completely distributive lattice dual to the same SFP-domain. As a consequence soundness and completeness of the infinitary logic is obtained for the class of finitary transition systems. A corollary of this result is that the same holds for the infinitary Hennessy-Milner logic.

1 Introduction

Complete partial orders were originally introduced as a mathematical structure to model computation [Sco70], in particular as domains for denotational semantics [SS71]. Successively, Scott's presentation of domains as information systems [Sco82] suggested a connection between denotational semantics and logics of programs. Based on the fundamental insight of Smyth [Smy83b] that a topological space may be seen as a 'data type' with the open sets as 'observable predicates', and functions between topological spaces as 'computations', Abramsky [Abr87, Abr91a], Zhang [Zha91] and Vickers [Vic89, AV93] developed a propositional program logic from a denotational semantics.

Abramsky [Abr87, Abr91a] uses Stone duality to relate two views of SFP-domains (a special kind of complete partial orders): one in terms of logic theories and one in terms of semantic models. Abramsky's starting point is that for an algebraic cpo P, its compact elements completely determine P, whereas for a logic the Lindenbaum algebra provides a model from which the logic can be recovered.

If P is an SFP-domain, then the collection $\mathcal{KO}(P)$ of all Scott compact open subsets of P ordered by subset inclusion forms a distributive lattice. The distributive lattice $\mathcal{KO}(P)$ can be viewed as the Lindenbaum algebra of a logic.

Conversely, every logic such that its Lindenbaum algebra is a distributive lattice L gives rise to a spectral space by taking the collection of all prime filters of L as points together with the filter topology [Joh82]. Spectral spaces include SFP-domains when taken with the Scott topology.

Abramsky [Abr87] gives a duality for SFP domains that can be built up in a modular way. He considers a number of basic constructors of domain theory, including lift, coalesced and separated sum, products, function space, Hoare, Smyth and Plotkin powerdomains, and recursion. Using the duality he shows that these constructors can be applied to Lindenbaum algebra's dual to SFP-domains, and hence can be used to generate logics for constructors applied to SFP-domains. Abramsky's theory applies therefore to all SFP-domains freely generated by the constructors.

Although mathematically very attractive, the logics of compact opens considered by Abramsky are weak in expressive power, and inadequate as a general specification formalism according to [Abr87]. What is needed is a language, with an accompanying semantic framework, which permits to go beyond compact open sets. In particular, there is the need for an infinitary logic with also infinite disjunctions and infinite conjunctions.

Since the spaces considered by Abramsky are spectral, the introduction of infinite disjunctions does not require a major adjustment of the semantic framework: we can consider the whole frame of open sets which is free over the distributive lattice of compact opens [Joh82].

The addition of infinite conjunctions is more difficult because it requires new mathematical tools. In this paper we give a new characterization of sober spaces in terms of the completely distributive lattice of saturated sets. This result allows us to freely extend the finitary logic of compact opens to the infinitary logic of saturated sets. The extension is conservative in the sense that the topological space represented by a finitary logic coincides with the one represented by its infinitary extension. The techniques involved are general and can be applied to every logic based on a topological interpretation.

As an application we treat Abramsky's domain logic for labelled transition systems with divergence [Abr91b]. Abramsky's domain logic for transition systems is equivalent to the Hennessy-Milner logic in the infinitary case, and hence it characterizes bisimulation for every transition system. However in the finitary case it is more satisfactory than the Hennessy-Milner logic in the sense that it characterizes a finitary preorder (the finitary observable part of bisimulation) for every transition system. Abramsky's infinitary logic can be used to characterize the class of transition systems for which the bisimulation preorders are algebraic, in the sense that they coincide with the finitary preorders. These transition systems are called finitary and satisfy two axiom schemas: one about bounded nondeterminism and another about finite approximation.

We prove soundness and completeness of the infinitary logic for the class of all finitary transition systems. The same completeness result holds also for the infinitary Hennessy-Milner logic because the latter is equivalent to Abramsky's infinitary logic. On the way to prove our completeness result, we also show soundness and completeness of Abramsky's logic with infinite disjunctions for the class of compactly branching transition systems.

2 A characterization of sober spaces

The purpose of this section is to derive a new characterization of sober spaces which will be given in Theorem 7 below. First we introduce the category of observation frames and relate it to the category of ordinary frames.

A lattice L is called *distributive* if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

for all a, b and c in L. The class of all distributive lattices together with functions preserving both finite meets and finite joins defines a category, denoted by **DLat**. If the lattice L has join for arbitrary subsets, and not just finite ones, and it satisfies the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

for all $a \in L$ and all subsets $S \subseteq L$ then it is a *frame*. Frames with functions preserving arbitrary joins and finite meets form a category called **Frm**. There is an obvious forgetful functor from **Frm** to **DLat** which has a left adjoint [Joh82].

A complete lattice L is *completely distributive* if, for all sets $\mathcal{A}$ of subsets of L,

$$\bigwedge \{\bigvee S \mid S \in \mathcal{A}\} = \bigvee \{\bigwedge f(\mathcal{A}) \mid f \in \Phi(\mathcal{A})\},$$

where $f(\mathcal{A})$ denotes the set $\{f(S) \mid S \in \mathcal{A}\}$ and $\Phi(\mathcal{A})$ is the set of all functions $f\colon \mathcal{A} \to \bigcup \mathcal{A}$ such that $f(S) \in S$ for all $S \in \mathcal{A}$. Completely distributive lattices with functions preserving both arbitrary meets and arbitrary joins form a category, denoted by **CDL**. Clearly every completely distributive lattice is a frame.

Theorem 1. *The forgetful functor* $\mathbf{CDL} \to \mathbf{Frm}$ *has a left adjoint denoted by* $\bar{(\cdot)}\colon \mathbf{Frm} \to \mathbf{CDL}$.

Proof. Immediate because the above categories are algebraic [Joh82, Bon96]. □

For a frame F let $\zeta_F\colon F \to \bar{F}$ be the unit of the above adjunction. Then ζ_F is a frame morphism and every element of the completely distributive lattice $\bar{F}$ is the meet of elements in $\zeta_F(F)$, that is, $\bar{F}$ is order generated by $\zeta_F(F)$. In general we call a map with these properties an observation frame.

Definition 2. An *observation frame* is a frame morphism $\alpha\colon F \to L$ between a frame F and a completely distributive lattice L such that, for every $q \in L$,

$$q = \bigwedge \{\alpha(x) \mid x \in F \text{ and } q \leq \alpha(x)\}.$$

Observation frames can be organized into a category, denoted by **OFrm**, with arrows defined as follows. A morphism between two observation frames $\alpha\colon F \to L$ and $\beta\colon G \to H$ is a pair $\langle f, g\rangle$ consisting of a frame morphism $f\colon F \to$

G and a complete distributive lattice morphism $g\colon L \to H$ such that $g \circ \alpha = \beta \circ f$ [BJK95, Bon96].

There is a functor $Dom\colon \mathbf{OFrm} \to \mathbf{Frm}$ mapping an observation frame $\alpha\colon F \to L$ to the frame $Dom(\alpha) = F$ and a morphism $\langle f, g\rangle$ in $\mathbf{OFrm}$ to the frame morphism $Dom(\langle f, g\rangle) = f$.

Theorem 3. *The functor $Dom\colon \mathbf{OFrm} \to \mathbf{Frm}$ has a left adjoint.*

Proof. Let F be a frame and consider the observation frame $\zeta_F\colon F \to \bar{F}$ defined as the unit of the adjunction given in Theorem 1. The identity function $id_F\colon F \to Dom(\zeta_F)$ is clearly a frame morphism. Moreover for every other observation frame $\beta\colon G \to H$ and frame morphism $f\colon F \to Dom(\beta)$ by Theorem 1 there exists a unique morphism $g\colon \bar{F} \to H$ such that $g \circ \zeta_F = \beta \circ f$. Hence $\langle f, g\rangle$ is the unique morphism in $\mathbf{OFrm}$ from ζ_F to β such that $Dom(\langle f, g\rangle) \circ id_F = f$. □

As a consequence of the above theorem we have, for a frame F, the following natural isomorphism:

$$\mathbf{Frm}(F, 2) = \mathbf{Frm}(F, Dom(\mathbf{2})) \cong \mathbf{OFrm}(\zeta_F, \mathbf{2}), \tag{1}$$

where $2 = \{\bot, \top\}$ is the two point completely distributive lattice (with $\bot \leq \top$) and $\mathbf{2}$ is the identity function on 2 (notice that $\mathbf{2}$ is an observation frame).

Recall that for a frame F, the set $\mathbf{Frm}(F, 2)$ is isomorphic to the set of completely prime filters of F [Vic89, Proposition 5.4.7]. Similarly, for an observation frame $\alpha\colon F \to L$, the set $\mathbf{OFrm}(\alpha, \mathbf{2})$ is isomorphic to the set of all completely prime M-filters of α [BJK95], where a filter $\mathcal{F}$ of F is an *M-filter* of the observation frame α if, for all $x \in F$,

$$\bigwedge \alpha(\mathcal{F}) \leq \alpha(x) \Rightarrow x \in \mathcal{F}.$$

We have therefore the following corollary of Theorem 3 and Equation (1).

Corollary 4. *The collection of all completely prime filters of a frame F coincides with the collection of all completely prime M-filters of the free observation frame $\zeta_F\colon F \to \bar{F}$.* □

Observation frames were introduced in [BJK95] in order to represent abstractly topological spaces: if X is a topological space, the inclusion $\mathcal{O}(X) \hookrightarrow \mathcal{Q}(X)$ mapping the frame of open sets into the completely distributive lattice of the saturated subsets of X forms an observation frame. We denote it by $\Omega(X)$. Moreover, if $f\colon X \to Y$ is a continuous function between spaces X and Y (i.e. a map in the category of topological spaces $\mathbf{Sp}$) then

$$\Omega(f) = \langle f^{-1}\colon \mathcal{O}(Y) \to \mathcal{O}(X), f^{-1}\colon \mathcal{Q}(Y) \to \mathcal{Q}(X)\rangle$$

is a morphism in $\mathbf{OFrm}$ between $\Omega(Y)$ and $\Omega(X)$. Thus we have a functor $\Omega\colon \mathbf{Sp} \to \mathbf{OFrm}^{op}$.

Next we show that Ω has a right adjoint. For an observation frame $\alpha\colon F \to L$ we denote by $Pt(\alpha)$ the topological space given by the set $CPMF(\alpha)$ of all

completely prime M-filters of α together with a topology with open sets defined, for every $x \in F$, by

$$\triangle(x) = \{\mathcal{F} \in CPMF(\alpha) \mid x \in \mathcal{F}\}.$$

Theorem 5. *The assignment $\alpha \mapsto Pt(\alpha)$, where $\alpha{:}F \to L$ is an observation frame, can be extended to a functor from $\mathbf{OFrm}^{op}$ to $\mathbf{Sp}$ which is right adjoint of Ω. The unit of the adjunction is given by the assignment $x \mapsto \{o \in \mathcal{O}(X) \mid x \in o\}$.*

Proof. See Theorem 3.23 in [BJK95]. □

The above adjunction restricts to an equivalence between the full subcategories $\mathbf{Sp}_0$ of $\mathcal{T}_0$ spaces and **SOFrm** of spatial observation frames [BJK95]. As a consequence, a space X is $\mathcal{T}_0$ if and only if the unit of the above adjunction determines an isomorphism in $\mathbf{Sp}$ between X and $Pt(\Omega(X))$.

Topological spaces can also be represented abstractly by considering only the frame of open sets: there is a functor $\mathcal{O}(-){:}\,\mathbf{Sp} \to \mathbf{Frm}^{op}$ which maps every topological space to its lattice of open sets and every continuous function to its inverse restricted to the open sets. Conversely, given a frame F, we can construct a topological space $Pt_\omega(F)$ by taking the set $CPF(F)$ of all completely prime filters of F with the filter topology. The space $Pt_\omega(F)$ is sober [Joh82], where a space X is said to be *sober* if the assignment

$$x \mapsto \{o \in \mathcal{O}(X) \mid x \in o\}$$

defines an isomorphism between X and $CPF(\mathcal{O}(X))$. The full subcategory of $\mathbf{Sp}$ with as objects sober spaces is denoted by **Sob**. The following proposition is standard, and can be found in [Joh82].

Proposition 6. *The assignment $F \mapsto Pt_\omega(F)$ defines a functor $\mathbf{Frm}^{op} \to \mathbf{Sp}$ which is a right adjoint of $\mathcal{O}(-){:}\,\mathbf{Sp} \to \mathbf{Frm}^{op}$. Furthermore we have that*

1. *the adjunction restricts to a duality between* **SFrm** *and* **Sob**;
2. *the inclusion $\mathbf{Sob} \hookrightarrow \mathbf{Sp}_0$ has left adjoint $Pt_\omega(\mathcal{O}(-))$.* □

The *soberification* of a topological space X is the sober space $Pt_\omega(\mathcal{O}(X))$. Notice that the process of soberification of a topological space X can be replaced by the process of constructing the topological space which best approximates the 'frame part' of the observation frame associated with X, that is,

$$Pt_\omega(\mathcal{O}(X)) = Pt_\omega(Dom(\Omega(X))).$$

Since adjoints are defined uniquely (up to natural isomorphisms), the above implies that the two squares below commute.

$$\begin{array}{ccc} \mathbf{Sp}_0 & \simeq & \mathbf{SOFrm}^{op} \\ {\scriptstyle Pt_\omega(\mathcal{O}(-))}\downarrow \dashv \uparrow{\scriptstyle i} & & {\scriptstyle Dom^{op}}\downarrow \dashv \uparrow{\scriptstyle (\bar{\cdot})^{op}} \\ \mathbf{Sob} & \simeq & \mathbf{SFrm}^{op} \end{array}$$

The functor Dom: **OFrm** $\to$ **Frm** can therefore be considered as the pointless soberification of an abstract topological space. Now we use the above results to derive the new characterization of sober spaces.

Theorem 7. *A $\mathcal{T}_0$ space X is sober if and only if the completely distributive lattice of saturated sets $\mathcal{Q}(X)$ is free over the frame of open sets $\mathcal{O}(X)$.*

Proof. Assume X is a sober space. By the commutativity of the above diagram it follows that the observation frames $\Omega(X)\colon \mathcal{O}(X) \to \mathcal{Q}(X)$ and $\zeta_{\mathcal{O}(X)}\colon \mathcal{O}(X) \to \mathcal{O}(\bar{X})$ are isomorphic in **OFrm**. Hence $\mathcal{O}(\bar{X})$ is isomorphic to $\mathcal{Q}(X)$ in **CDL**. But $\mathcal{O}(\bar{X})$ is the free completely distributive lattice over the frame $\mathcal{O}(X)$, by Theorem 1.

For the converse, assume X is a $\mathcal{T}_0$ space and $\mathcal{Q}(X)$ is the free completely distributive lattice over the frame $\mathcal{O}(X)$. Then the set of all completely prime M-filters of $\Omega(X)$ coincides with the set of all completely prime M-filters of $\zeta_{\mathcal{O}(X)}$, which, by Corollary 4, coincides with the set of all completely prime filters of $\mathcal{O}(X)$. Since X is a $\mathcal{T}_0$ space, the assignment

$$x \mapsto \{o \in \mathcal{O}(X) \mid x \in o\}$$

is an isomorphism between X and $CPMF(\Omega(X))$. But

$$CPF(\mathcal{O}(X)) = CPMF(\Omega(X)),$$

hence X is a sober space. □

3 Domain theory in infinitary logical form

In this section we briefly discuss Abramsky's framework [Abr87, Abr91a] for connecting denotational semantics and program logic, and explain how the results of the previous section can be used to extend it.

Abramsky considers a typed language together with a denotational interpretation which takes values in the category **SFP** of SFP-domains. The language has several type constructors which are interpreted denotationally as the ordinary domain theoretical constructors, such as products, coproducts, function space and powerdomains. A second logical interpretation associates to each type of the language a propositional theory. Each theory has axioms and rules which enforce a distributive lattice structure with finite meets and finite joins. Moreover, for each type constructor there is a corresponding constructor between propositional theories.

The logical interpretation and the denotational interpretation are connected as follows. For any SFP-domain, the compact open sets of its Scott topology form a distributive lattice when ordered by subset inclusion. Conversely, every propositional theory with a structure of a distributive lattice gives rise to a complete partial order by taking the set of prime filters of its Lindenbaum algebra (which

is a distributive lattice) together with the inclusion order [Joh82]. The main result of Abramsky is that for each type of the language, its logical interpretation is isomorphic as lattice to the Scott compact open subsets of its denotational interpretation. Conversely, its denotational interpretation is isomorphic as complete partial order to the complete partial order of the prime filters of its logical interpretation.

This implies that an element of an SFP-domain can be considered equivalent to the set of all properties satisfied by that element, which therefore gives a logical characterization of it. Even more, the order of the SFP-domain can be characterized in terms of the properties satisfied by the elements, that is, an element is smaller or equal to another one if and only if every property satisfied by the first element is also satisfied by the other one.

It is important to stress here that the propositional theories used by Abramsky for the logical interpretation of the type language are finite. Infinitary propositional theories can be considered as well without any major adjustment of the framework. This is a consequence of the results of the previous section. We first describe the case of infinitary propositional geometric theories, with infinite joins and finite meets, and then the case of infinitary propositional theories, with both infinite joins and infinite meets.

Consider a new logical interpretation for each type of the language which extends the previous one as follows. For every type there is an associated propositional theory with axioms and rules enforcing a structure of a frame, with infinite joins and finite meets, and such that every element of the theory can be proved equivalent to an (infinite) join of elements of the finitary theory used by Abramsky. In other words, the Lindenbaum algebra of the infinitary propositional geometric theory is the free frame over the distributive lattice of the finitary theory considered by Abramsky. Since Scott open subsets of an SFP-domain ordered by subset inclusion form the free frame over the distributive lattice of the Scott compact subsets [Joh82], it follows that for each type of the language, its new logical interpretation is isomorphic as lattice to the Scott open subsets of its denotational interpretation. Conversely, its denotational interpretation is isomorphic as SFP-domain to the domain of completely prime filters of its logical interpretation. Hence the connection of Abramsky between finite propositional theories and SFP-domains extends smoothly if we consider propositional theories with infinite joins and finite meets.

The above extension is standard. Using the characterization of sober spaces given in the previous section we can extend the logical interpretation of the type language even further. We can associate to every type an infinitary propositional theory with axioms and rules enforcing a structure of a completely distributive lattice, with both infinite joins and infinite meets, and such that it freely extends Abramsky's interpretation. This means that the Lindenbaum algebra of the infinitary propositional theory is the free completely distributive lattice over the distributive lattice of the Lindenbaum algebra induced by Abramsky's original logical interpretation. Since an SFP-domain is a sober space when taken with the Scott topology, its saturated sets form the free completely distributive lat-

tice over the frame of the open sets and hence over the distributive lattice of the Scott compact open subsets. It follows that for each type in the language, its infinitary logical interpretation is isomorphic as lattice to the saturated sets of its denotational interpretation, while its denotational interpretation is isomorphic as SFP-domain to the domain of completely prime M-filters of the observation frame induced by its logical interpretation (the map which embeds the restricted theory with infinite joins and finite meet into the infinitary one).

4 Domain logic for transition systems

As an application of the techniques discussed above, we treat Abramsky's domain logic for labelled transition systems with divergence [Abr91b]. We begin by recalling some notions on labelled transition systems.

Definition 8. A *labelled transition system* with divergence $\langle P, Act, \longrightarrow, \Uparrow\rangle$ is defined by a set P of processes, a set Act of atomic actions, a transition relation $\longrightarrow \subseteq P \times Act \times P$, and a predicate $\Uparrow$ on P. The predicate $\Uparrow$ is called the *divergence predicate*. The *convergence predicate* $\Downarrow$ on P is defined to be the complement of the divergence predicate, that is $\Downarrow = P \setminus \Uparrow$. We use $p \Uparrow$ and $p \Downarrow$ to denote that the process p diverges and converges, respectively.

Transition systems can be used for modelling computations of programming languages [Plo81b] and to identify processes with the same observable behaviour. One of the well-known behavioral equivalences on processes is bisimulation [Mil80, Par81].

Definition 9. Given a transition system $\langle P, Act, \longrightarrow, \Uparrow\rangle$, a relation $R \subseteq P \times P$ is called a *partial bisimulation* whenever, if $\langle p, q\rangle \in R$ then for all $a \in Act$

1. $p \xrightarrow{a} p' \Rightarrow \exists q' \in P$: $q \xrightarrow{a} q'$ and $\langle p', q'\rangle \in R$;
2. $p \Downarrow \Rightarrow q \Downarrow$ and $(q \xrightarrow{a} q' \Rightarrow \exists p' \in P$: $p \xrightarrow{a} p'$ and $\langle p', q'\rangle \in R)$.

We write $p \lesssim^B q$ if there exists a partial bisimulation R with $\langle p, q\rangle \in R$.

Partial bisimulations can also be described in terms of iteration [Par81], but in general one needs to consider a non-countable sequence of relations (in the complete lattice $\mathcal{P}(P \times P)$ ordered by subset inclusion) approximating $\lesssim^B$. By considering only countable approximants of $\lesssim^B$ one obtains the so-called *observable equivalence* $\lesssim^\omega = \bigcap_\omega \lesssim^n$ [Mil80], where

- $\lesssim^0 = P \times P$, and
- $p \lesssim^{n+1} q$ if and only if for all $a \in Act$
  1. $p \xrightarrow{a} p' \Rightarrow \exists q' \in P$: $q \xrightarrow{a} q'$ and $p' \lesssim^n q'$;
  2. $p \Downarrow \Rightarrow q \Downarrow$ and $(q \xrightarrow{a} q' \Rightarrow \exists p' \in P$: $p \xrightarrow{a} p'$ and $p' \lesssim^n q')$.

In general for a transition system T, $\lesssim^B \subseteq \lesssim^\omega$. However, if T is image-finite then the two notions coincide [HM85].

A particular example of a transition system is given by the collection of all (finite) *synchronization trees* over an alphabet Act of actions. Define the set $(t \in)ST(Act)$ of finitary synchronization trees over Act by

$$t ::= \Sigma_I\, a_i t_i \mid \Sigma_I\, a_i t_i + \Omega,$$

where I is a finite index set, and all the a_i's are actions in Act for $i \in I$. The set of all finitary synchronization trees can be turned into a transition system $ST(Act) = \langle ST(Act), Act, \longrightarrow, \Uparrow \rangle$, where

- $t \Uparrow$ if and only if Ω is included as a summand of t, and
- $t \xrightarrow{a_i} t_i$ for each summand $a_i t_i$ of t.

Synchronization trees can be used to define a finitary preorder on processes of more general transition systems [Gue81].

Definition 10. For a transition system $\langle P, Act, \longrightarrow, \Uparrow \rangle$ define the *finitary preorder* $\lesssim^F \subseteq P \times P$ by

$$p \lesssim^F q \text{ if and only if } \forall t \in ST(Act)\colon t \lesssim^B p \Rightarrow t \lesssim^B q.$$

Since finite synchronization trees are a model for finite processes, the finitary preorder can be considered as the finite observable part of partial bisimulation. For every transition system T, it holds that

$$\lesssim^B \subseteq \lesssim^\omega \subseteq \lesssim^F .$$

In general, these inclusions are strict [Abr91b, pag. 191].

Another example of a transition system is given by the SFP-domain $\mathcal{D}$ obtained as the initial (and final) solution in the category **SFP** of the recursive domain equation

$$X \cong (\mathbf{1})_\perp \oplus \mathcal{P}_c^{co}\left(\sum_{a \in Act} X\right),$$

where $\mathbf{1}$ is the one-point cpo, Act is a countable set of actions, $(-)_\perp$ is the lift, $\oplus$ is the coalesced sum, $\sum_{a \in Act}$ is the countable separated sum, and $\mathcal{P}_c^{co}(-)$ is the Plotkin powerdomain. Below we will omit the isomorphism pair relating the left and the right hand side of the solution $\mathcal{D}$ of the above domain equation. The SFP-domain $\mathcal{D}$ can be seen as the transition system $\langle \mathcal{D}, Act, \longrightarrow, \Uparrow \rangle$ where

- $d \xrightarrow{a} d'$ if and only if $\langle a, d' \rangle \in d$ and
- $d \Uparrow$ if and only if $\perp \in d$.

The SFP-domain $\mathcal{D}$, seen as a transition system, plays the role of canonical model for the Abramsky's logic for transition systems. Furthermore the domain $\mathcal{D}$ can be used as a semantic domain for every transition system modulo the equivalence generated by the finitary preorder [Abr91b].

Like the Hennessy-Milner logic [HM85], the idea of Abramsky's infinitary logic $\mathcal{L}_{\infty,\infty}$ for transition systems [Abr91b] is to obtain a suitable characterization of partial bisimulation in terms of a notion of property of processes: $p \lesssim^B q$ if and only if every property satisfied by p is also satisfied by q. However, the finitary restriction of Abramsky's logic differs from the finitary Hennessy-Milner logic in the sense that it characterizes the finitary observable part of partial bisimulation for all transition systems.

Definition 11. Let $(a \in) Act$ be a set of actions. The language $\mathcal{L}_{\infty,\infty}$ over Act has two sorts: π (processes) and k (capabilities). We write $(\phi \in)\mathcal{L}^{\pi}_{\infty,\infty}$ for the class of formulae of sort π, and $(\psi \in)\mathcal{L}^{k}_{\infty,\infty}$ for the class of formulae of sort k, which are defined inductively as follows:

$$\phi ::= \bigvee_I \phi_i \mid \bigwedge_I \phi_i \mid \Box\psi \mid \Diamond\psi$$
$$\psi ::= \bigvee_I \psi_i \mid \bigwedge_I \psi_i \mid a(\phi),$$

where I is an arbitrary index set. If $I = \emptyset$ then we write tt for $\bigwedge_I \phi_i$ and $\bigwedge_I \psi_i$, and we write ff for $\bigvee_I \phi_i$ and $\bigvee_I \psi_i$.

Before we interpret the language $\mathcal{L}_{\infty,\infty}$ we need the following definitions. For a transition system $\langle P, Act, \longrightarrow, \Uparrow\rangle$ define the set Cap of *capabilities* by

$$Cap = \{\bot\} \cup (Act \times P).$$

The set of capabilities of a process $p \in P$ is given by

$$C(p) = \{\bot \mid p \Uparrow\} \cup \{\langle a, q\rangle \mid p \xrightarrow{a} q\}.$$

For a transition system $T = \langle P, Act, \longrightarrow, \Uparrow\rangle$, we interpret the language $\mathcal{L}_{\infty,\infty}$ by means of *satisfaction relations* $\models_\pi \subseteq P \times \mathcal{L}^{\pi}_{\infty,\infty}$ and $\models_k \subseteq Cap \times \mathcal{L}^{k}_{\infty,\infty}$ defined as follows:

$$\begin{array}{lcl}
p \models_\pi \bigvee_I \phi_i & \iff & \exists i \in I\colon\ p \models_\pi \phi_i \\
p \models_\pi \bigwedge_I \phi_i & \iff & \forall i \in I\colon\ p \models_\pi \phi_i \\
p \models_\pi \Box\phi & \iff & p \Downarrow \text{ and } \forall c \in C(p)\colon\ c \models_k \phi \\
p \models_\pi \Diamond\phi & \iff & \exists c \in C(p)\colon\ c \models_k \phi
\end{array}$$

$$\begin{array}{lcl}
c \models_k \bigvee_I \phi_i & \iff & \exists i \in I\colon\ c \models_k \phi_i \\
c \models_k \bigwedge_I \phi_i & \iff & \forall i \in I\colon\ c \models_k \phi_i \\
c \models_k a(\phi) & \iff & c = \langle a, q\rangle \text{ and } q \models_\pi \phi.
\end{array}$$

For a transition system $T = \langle P, Act, \longrightarrow, \Uparrow\rangle$ and formula ϕ of $\mathcal{L}^{\pi}_{\infty,\infty}$ we write $[\![\phi]\!]^{\pi}_T$ for $\{p \in P \mid p \models_\pi \phi\}$. *Assertions* A over the language $\mathcal{L}^{\sigma}_{\infty,\infty}$ are of the form $\phi \leq_\sigma \psi$ or $\phi =_\sigma \psi$ for σ in $\{\pi, k\}$ with ϕ and ψ in $\mathcal{L}^{\sigma}_{\infty,\infty}$. The satisfaction relation between transition systems T and assertions is defined by

$$\begin{array}{lcl}
T \models \phi \leq_\pi \psi & \iff & \forall p \in P\colon\ p \models_\pi \phi \text{ implies } p \models_\pi \psi \\
T \models \phi =_\pi \psi & \iff & \forall p \in P\colon\ p \models_\pi \phi \text{ if and only if } p \models_\pi \psi
\end{array}$$

$$\begin{array}{lcl}
T \models \phi \leq_k \psi & \iff & \forall c \in Cap\colon\ c \models_k \phi \text{ implies } c \models_k \psi \\
T \models \phi =_k \psi & \iff & \forall c \in Cap\colon\ c \models_k \phi \text{ if and only if } c \models_k \psi.
\end{array}$$

As usual, the satisfaction relation can be extended to classes of transition systems $\mathcal{T}$ by

$$\mathcal{T} \models A \iff \forall\ T \in \mathcal{T}\colon\ T \models A.$$

If $\mathcal{T}$ is the class of all transition systems then we simply write $\models A$.

Let $\mathcal{L}_{\omega,\omega}$ be the sub-language of $\mathcal{L}_{\infty,\infty}$ obtained by the restriction to finite conjunctions and finite disjunctions.

Theorem 12. *For a transition system $\langle P, Act, \longrightarrow, \Uparrow\rangle$ and p, q in P,*

1. *$p \lesssim^B q$ if and only if $\forall \phi \in \mathcal{L}^{\pi}_{\infty,\infty}\colon p \models \phi \Rightarrow q \models \phi$;*
2. *$p \lesssim^F q$ if and only if $\forall \phi \in \mathcal{L}^{\pi}_{\omega,\omega}\colon p \models \phi \Rightarrow q \models \phi$.*

Proof. See Theorems 5.6 and 5.8 in [Abr91b]. □

Next we present a proof system for assertions over $\mathcal{L}_{\infty,\infty}$. (We omit the sort subscripts.) The following *logical axioms* give to the language the structure of a large completely distributive lattice.

$(\leq -ref)\quad \phi \leq \phi$

$(\leq -trans)\quad \dfrac{\phi \leq \psi \text{ and } \psi \leq \chi}{\phi \leq \chi}$

$(= -I)\quad \dfrac{\phi \leq \psi \text{ and } \psi \leq \phi}{\phi = \psi}$ $\qquad$ $(= -E)\quad \dfrac{\phi = \psi}{\phi \leq \psi \text{ and } \psi \leq \phi}$

$(\wedge - I)\quad \dfrac{\{\phi \leq \psi_i\}_{i\in I}}{\phi \leq \bigwedge\limits_I \psi_i}$ $\qquad$ $(\wedge - E)\quad \bigwedge_I \phi_i \leq \phi_k \qquad (k \in I)$

$(\vee - I)\quad \dfrac{\{\phi_i \leq \psi\}_{i\in I}}{\bigvee\limits_I \phi_i \leq \psi}$ $\qquad$ $(\vee - E)\quad \phi_k \leq \bigvee_I \phi_i \qquad (k \in I)$

$(\wedge - dist)\quad \bigwedge_I \bigvee_{J_i} \phi_{i,j} = \bigvee_{f\in\Phi(\{J_i \mid i\in I\})} \bigwedge_I \phi_{i,f(i)}.$

The following *modal axioms* relate constructors with the logical structure.

$(a- \leq)\quad \dfrac{\phi \leq \psi}{a(\phi) \leq a(\psi)}$

$(a - \wedge)$ (i) $a(\bigwedge_I \phi_i) = \bigwedge_I a(\phi_i) \qquad (I \neq \emptyset)$

(ii) $a(\phi) \wedge b(\psi) = F \qquad (a \neq b)$

$(a - \vee)\quad a(\bigvee_I \phi_i) = \bigvee_I a(\phi_i)$

$$(\Box -\leq)\quad \frac{\phi \leq \psi}{\Box\phi \leq \Box\psi}$$

$$(\Box -\wedge)\quad \Box \bigwedge_I \phi_i = \bigwedge_I \Box\phi_i \qquad (I \neq \emptyset)$$

$$(\Box -\vee)\quad \Box(\phi \vee \psi) \leq \Box\phi \vee \Diamond\psi$$

$$(\Diamond -\leq)\quad \frac{\phi \leq \psi}{\Diamond\phi \leq \Diamond\psi}$$

$$(\Diamond -\wedge)\quad \Box\phi \wedge \Diamond\psi \leq \Diamond(\phi \wedge \psi)$$

$$(\Diamond -\vee)\quad \Diamond \bigvee_I \phi_i = \bigvee_I \Diamond\phi_i$$

We write $\mathcal{L}_{\infty,\infty} \vdash A$ if the assertion A of $\mathcal{L}_{\infty,\infty}$ is derivable from the above axioms and rules.

Theorem 13. *(Soundness) If $\mathcal{L}_{\infty,\infty} \vdash A$ then $\models A$.*

Proof. See Theorem 4.2 in [Abr91b]. □

Next we turn to the finitary logic $\mathcal{L}_{\omega,\omega}$ in order to prove the reverse of the above result for the class of all transition systems.

Let $\mathcal{LA}^{\pi}_{\omega,\omega}$ be the *Lindenbaum algebra* of $\mathcal{L}^{\pi}_{\omega,\omega}$, and let $[\phi]$ denote the set of all formulae provable equivalent in $\mathcal{L}^{\pi}_{\omega,\omega}$ to $\phi \in \mathcal{L}^{\pi}_{\omega,\omega}$. The following fundamental result shows that the finitary logic $\mathcal{L}^{\pi}_{\omega,\omega}$ does indeed correspond exactly to the SFP-domain $\mathcal{D}$ taken with the Scott topology.

Theorem 14. *Let $\mathcal{KO}(\mathcal{D})$ be the distributive lattice of Scott compact open sets of $\mathcal{D}$ ordered by subset inclusion. The function $\gamma\colon \mathcal{LA}^{\pi}_{\omega,\omega} \to \mathcal{KO}(\mathcal{D})$ defined, for ϕ in $\mathcal{L}^{\pi}_{\omega,\omega}$, by*

$$\gamma([\phi]) = [\![\phi]\!]^{\pi}_{\mathcal{D}}$$

is a well-defined order isomorphism.

Proof. See Theorem 4.3 in [Abr91b]. □

The proof of the spatiality of the distributive lattice $\mathcal{LA}^{\pi}_{\omega,\omega}$ is equivalent to (strong) completeness of the underlying logical system.

Theorem 15. *(Completeness) Let $\mathcal{T}$ be any class of transition systems containing $\mathcal{D}$. For ϕ_1 and ϕ_2 in $\mathcal{L}^{\pi}_{\omega,\omega}$, $\mathcal{T} \models \phi_1 \leq \phi_2$ if and only if $\mathcal{L}^{\pi}_{\omega,\omega} \vdash \phi_1 \leq \phi_2$.*

Proof. For ϕ_1 and ϕ_2 in $\mathcal{L}^{\pi}_{\omega,\omega}$ we have,

$$\begin{aligned}
\mathcal{D} \models_{\pi} \phi_1 \leq_{\pi} \phi_2 &\iff [\![\phi_1]\!]^{\pi}_{\mathcal{D}} \subseteq [\![\phi_2]\!]^{\pi}_{\mathcal{D}} \\
&\iff \gamma([\phi_1]) \subseteq \gamma([\phi_2]) \quad \text{[definition of } \gamma\text{]} \\
&\iff [\phi_1] \leq [\phi_2] \quad [\gamma \text{ is an order isomorphism}] \\
&\iff \mathcal{L}^{\pi}_{\omega,\omega} \vdash \phi_1 \leq_{\pi} \phi_2. \quad \text{[definition of } \mathcal{LA}^{\pi}_{\omega,\omega}\text{]}
\end{aligned}$$

□

We conclude this section by showing that the SFP-domain $\mathcal{D}$ can be used as semantic domain for all transition systems. Let $T = \langle P, Act, \longrightarrow, \Uparrow \rangle$ be a transition system and let $p \in P$. The set

$$TS(p) = \{[\phi] \in \mathcal{LA}^{\pi}_{\omega,\omega} \mid p \models_{\pi} \phi\}$$

is a prime filter of the distributive lattice $\mathcal{LA}^{\pi}_{\omega,\omega}$. Hence, by Theorem 14, it corresponds uniquely to an element in $\mathcal{D}$. Therefore, the assignment $p \mapsto TS(p)$ defines a function $TS[\![\cdot]\!]\colon P \to \mathcal{D}$ which is unique among all functions $f\colon P \to \mathcal{D}$ such that

$$p \models_{\pi} \phi \text{ if and only if } f(p) \models_{\pi} \phi,$$

for all $p \in P$ and $\phi \in \mathcal{L}^{\pi}_{\omega,\omega}$ [Abr91a, Theorem 5.21]. By the characterization Theorem 12, it follows that p and $TS[\![p]\!]$ are equivalent in the finitary preorder $\lesssim^{F}$. Hence the function $TS[\![\cdot]\!]\colon P \to \mathcal{D}$ can be regarded as a syntax-free semantics which is universal because it is defined for every transition system.

5 Compactly branching transition systems

Theorem 15 gives a completeness result for $\mathcal{L}_{\omega,\omega}$. In this section we derive a completeness result for $\mathcal{L}_{\omega,\infty}$, the sub-language of $\mathcal{L}_{\infty,\infty}$ which allows infinite disjunctions but has only finite conjunctions. It is possible to express useful properties in this language that cannot be expressed in $\mathcal{L}_{\omega,\omega}$. Consider properties of a transition system $\langle P, Act, \longrightarrow, \Uparrow \rangle$ like 'the process p converges', 'every a-path starting from p is finite', or 'along every a-path starting from p eventually ψ holds'. The finitary language $\mathcal{L}^{\pi}_{\omega,\omega}$ is too weak to formalize these properties, which however can be expressed in the infinitary language $\mathcal{L}^{\pi}_{\omega,\infty}$ by

- $p \models_{\pi} \Box \bigvee_{a \in Act} a(tt)$;
- $p \models_{\pi} \bigvee_{n \in \omega} \phi_n$, where $\begin{cases} \phi_0 & = ff \text{ and} \\ \phi_{n+1} & = \Box(a(\phi_n) \vee \bigvee_{Act \setminus \{a\}} b(tt)); \end{cases}$
- $p \models_{\pi} \bigvee_{n \in \omega} \phi_n$, where $\begin{cases} \phi_0 & = ff \text{ and} \\ \phi_{n+1} & = \psi \vee (\Diamond a(tt) \wedge \Box(a(\phi_n) \vee \bigvee_{Act \setminus \{a\}} b(tt))). \end{cases}$

Adding expressive power to the finitary logic should not change our main motivation for its introduction: it should characterize the finitary observable part of partial bisimulation. We introduce the following scheme over $\mathcal{L}_{\omega,\infty}$ which restricts the class of transition systems and allows to write any formula in $\mathcal{L}_{\omega,\infty}$ as disjunctions (possibly infinite) of finitary formulae in $\mathcal{L}_{\omega,\omega}$:

$$(BN)\ \Box \bigvee_I \phi_i \leq \bigvee_{J \in Fin(I)} \Box \bigvee_J \phi_j \ (\phi_i \in \mathcal{L}_{\omega,\omega})$$

where $Fin(I)$ is the set of all finite subsets of I. The intuition behind the above axiom scheme is that of *bounded nondeterminism*. Indeed (BN) is equivalent to requiring that the $\Box$ operator is Scott continuous. Semantically, it corresponds to a statement of compactness (and hence of bounded non-determinism [Plo81a, Smy83b]): Let $T = \langle P, Act, \longrightarrow, \Uparrow \rangle$ be a transition system and let $\mathcal{O}(T)$ denote

the set of all $[\![\phi]\!]^\pi_T$ for ϕ in $\mathcal{L}^\pi_{\omega,\infty}$. Clearly, $\mathcal{O}(T)$ forms a topology on P. Transition systems together with a topology are introduced in the context of modal logic in [Esa74], where, in a restricted form, the direction from left to right of the next lemma is proved (the proof of the other direction can been found in [BK95]).

Lemma 16. *A transition system $T = \langle P, Act, \longrightarrow, \Uparrow\rangle$ satisfies (BN) if and only if for all $p \in P$ such that $p \Downarrow$ the set*

$$Br(p) = \{q \in P \mid \exists a \in Act\colon p \xrightarrow{a} q\}$$

is compact in the topology $\mathcal{O}(T)$.

Proof. Assume T satisfies (BN), take a $p \in P$ with $p \Downarrow$ and $Br(p) \subseteq \bigcup_I [\![\phi_i]\!]^\pi_T$. Then $p \models_\pi \Box \bigvee_I \phi_i$. Hence, by (BN), $p \models_\pi \bigvee_{J \in Fin(I)} \Box \bigvee_J \phi_j$, that is, $Br(p) \subseteq \bigcup_J [\![\phi_j]\!]^\pi_T$ for a finite subset J of I. Hence $Br(p)$ is compact in $\mathcal{O}(T)$.

Conversely, assume that if $p \Downarrow$ then the set $Br(p)$ is compact in the topology $\mathcal{O}(T)$, and let $p \models_\pi \Box \bigvee_I \phi_i$. Since p converges, $Br(p) \subseteq \bigcup_I [\![\phi_i]\!]^\pi_T$. But $Br(p)$ is compact, hence $Br(p) \subseteq \bigcup_J [\![\phi_j]\!]^\pi_T$ for some finite subset J of I. It follows that p satisfies (BN). □

A transition system is called *compactly branching* if it satisfies all instances of (BN). Every weakly finitely branching transition system is compactly branching [Abr91b]. Also, the transition system induced by the SFP-domain $\mathcal{D}$ is compactly branching (and not weakly finite branching).

Lemma 17. *For every formula ϕ in $\mathcal{L}^\pi_{\omega,\infty}$ there exist formulae $\phi_i \in \mathcal{L}_{\omega,\omega}$ with $i \in I$ such that $\mathcal{L}^\pi_{\omega,\infty} + (BN) \vdash \phi = \bigvee_I \phi_i$.*

Proof. Adapt Lemma 5.17 of [Abr91b]. □

The above lemma together with the soundness Theorem 13, the definition of the satisfaction relation, and the characterization Theorem 12, imply that for compactly branching transition systems $\langle P, Act, \longrightarrow, \Uparrow\rangle$ and processes p, q in P,

$$p \lesssim^F q \text{ if and only if } \forall \phi \in \mathcal{L}^\pi_{\omega,\infty}\colon p \models \phi \Rightarrow q \models \phi.$$

The next step is to prove the completeness of the logic $\mathcal{L}^\pi_{\omega,\infty}$ for the class of compactly branching transition systems. We proceed as for the finite case: let $\mathcal{LA}^\pi_{\omega,\infty}$ be the Lindenbaum algebra of $\mathcal{L}^\pi_{\omega,\infty}$ with as elements equivalences classes of formulae provable equivalent in $\mathcal{L}^\pi_{\omega,\infty} + (BN)$. The poset $\mathcal{LA}^\pi_{\omega,\infty}$ is a frame with meets and joins defined as expected.

Lemma 18. *The frame $\mathcal{LA}^\pi_{\omega,\infty}$ is free over the distributive lattice $\mathcal{LA}^\pi_{\omega,\omega}$.*

Proof. For every frame F and function $f\colon \mathcal{LA}^\pi_{\omega,\omega} \to F$ preserving finite meets and finite joins, define $h\colon \mathcal{LA}^\pi_{\omega,\infty} \to F$ by $h([\phi]) = \bigvee_I f([\phi_i])$ where, using Lemma 17,

$[\phi] = \bigvee_I [\phi_i]$ with ϕ_i in $\mathcal{L}^\pi_{\omega,\omega}$. By definition, h preserves joins and $h([\phi]) = f([\phi])$ for all ϕ in $\mathcal{L}^\pi_{\omega,\omega}$. Moreover, h preserves finite meets:

$$\begin{aligned}
h([\phi] \wedge [\phi']) &= h([\bigvee_I \phi_i] \wedge [\bigvee_J \phi'_j]) \quad \text{[Lemma 17]} \\
&= h(\bigvee_{I\times J} [\phi_i \wedge \phi'_j]) \quad \text{[distributivity]} \\
&= \bigvee_{I\times J} f([\phi_i \wedge \phi'_j]) \quad \text{[h preserves joins and commutativity]} \\
&= \bigvee_{I\times J} (f([\phi_i]) \wedge f([\phi'_j])) \quad \text{[f preserves meets]} \\
&= \bigvee_{I\times J} (h([\phi_i]) \wedge h([\phi'_j])) \quad \text{[commutativity]} \\
&= \bigvee_I h([\phi_i]) \wedge \bigvee_J h([\phi'_j]) \quad \text{[distributivity]} \\
&= h([\phi]) \wedge h([\phi']).
\end{aligned}$$

Hence h is the unique frame morphism such that $h \circ \iota = f$, where $\iota\colon \mathcal{LA}^\pi_{\omega,\omega} \to \mathcal{LA}^\pi_{\omega,\infty}$ is the obvious inclusion function. □

We can now draw an interesting consequence of the finitary axiom (BN).

Lemma 19. *The assignment $[\phi] \mapsto [\![\phi]\!]^\pi_{\mathcal{D}}$ defines a unique order isomorphism $\gamma^+\colon \mathcal{LA}^\pi_{\omega,\infty} \to \mathcal{O}(\mathcal{D})$ such that $\gamma^+([\phi]) = \gamma([\phi])$ for all ϕ in $\mathcal{L}^\pi_{\omega,\omega}$.*

Proof. Because $\mathcal{D}$ is an SFP-domain, when taken with its Scott topology it forms a spectral space. Hence the lattice of Scott open sets $\mathcal{O}(\mathcal{D})$ is the free frame over the distributive lattice of Scott compact open sets $\mathcal{KO}(\mathcal{D})$, which, by Lemma 14, is order isomorphic to the Lindenbaum algebra $\mathcal{LA}^\pi_{\omega,\omega}$. But $\mathcal{LA}^\pi_{\omega,\infty}$ is the free frame over the distributive lattice $\mathcal{LA}^\pi_{\omega,\omega}$ (Lemma 18), hence $\mathcal{O}(\mathcal{D})$ is order isomorphic to $\mathcal{LA}^\pi_{\omega,\infty}$. The isomorphism is given by the unique extension γ^+ of the function $\gamma\colon \mathcal{LA}^\pi_{\omega,\omega} \to \mathcal{KO}(\mathcal{D})$ given in Lemma 18. Using Lemma 17 and the soundness Theorem 13, it can be characterized by

$$\gamma^+([\phi]) = \bigcup_I \gamma([\phi_i]) = \bigcup_I [\![\phi_i]\!]^\pi_{\mathcal{D}} = [\![\bigvee_I \psi_i]\!]^\pi_{\mathcal{D}} = [\![\phi]\!]^\pi_{\mathcal{D}},$$

for all ϕ in $\mathcal{L}^\pi_{\omega,\infty}$. □

Soundness of the logical system associated to $\mathcal{L}^\pi_{\omega,\infty}$ including the scheme (BN) follows from Theorem 13 and the definition of compactly branching transition systems. In a way similar to the completeness Theorem 15, completeness follows from the duality Lemma 19.

Theorem 20. *(Completeness) Let $\mathcal{CB}$ be any class of compactly branching transition systems containing $\mathcal{D}$. For ϕ_1 and ϕ_2 in $\mathcal{L}^\pi_{\omega,\infty}$, $\mathcal{CB} \models \phi_1 \leq \phi_2$ if and only if $\mathcal{L}^\pi_{\omega,\infty} + (BN) \vdash \phi_1 \leq \phi_2$.* □

6 Finitary transition systems

The language $\mathcal{L}_{\omega,\infty}$ is more expressive than the finitary language $\mathcal{L}_{\omega,\omega}$. Next we consider the even more expressive language $\mathcal{L}_{\infty,\infty}$. For example, given a transition system $\langle P, Act, \longrightarrow, \Uparrow\rangle$ we can specify in $\mathcal{L}_{\infty,\infty}$ properties like 'there exists an infinite a-path starting from the process p', and 'at any point of any path starting from p an a-transition is always possible', respectively by

- $p \models_\pi \bigwedge_{n\in\omega} \phi_n$, where $\begin{cases} \phi_0 & = tt \text{ and} \\ \phi_{n+1} & = \Diamond a(\phi_n); \end{cases}$
- $p \models_\pi \bigwedge_{n\in\omega} \phi_n$, where $\begin{cases} \phi_0 & = tt \text{ and} \\ \phi_{n+1} & = \Diamond a(\phi_n) \wedge \bigwedge_{Act}(\Box b(\phi_n) \vee \bigvee_{Act\setminus\{b\}} c(tt)). \end{cases}$

By using observation frames we can now prove a completeness result for $\mathcal{L}_{\infty,\infty}$ following the same pattern as for the completeness result of $\mathcal{L}_{\omega,\infty}$.

We introduction two finitary axiom schemes over $\mathcal{L}_{\infty,\infty}$. These schemes will restrict the class of transition systems under consideration, and will allow us to write a formula in $\mathcal{L}_{\infty,\infty}$ as a conjunction of disjunctions of finitary formulae in $\mathcal{L}_{\omega,\omega}$:

$$(BN)\ \Box \bigvee_I \phi_i \leq \bigvee_{J\in Fin(I)} \Box \bigvee_J \phi_j \ (\phi_i \in \mathcal{L}_{\omega,\omega})$$
$$(FA)\ \bigwedge_{J\in Fin(I)} \Diamond \bigwedge_J \phi_j \leq \Diamond \bigwedge_I \phi_i \ (\phi_i \in \mathcal{L}_{\omega,\omega}),$$

where $Fin(I)$ is the set of all finite subsets of I. The axiom scheme (FA) is the dual of (BN). While the axiom (BN) is related to the width of a computation, the axiom (FA) is related to the length of it. The latter is analogous to the requirement that we cannot distinguish a set from its closure by means of compact open sets [Fin73] (thinking of each ϕ_i as a compact open set, or, equivalently, as a finite observable property). It can be understood as a notion of *finite approximation*.

For example, the transition system induced by the set of finite synchronization trees satisfies all instances of the two axiom schemes above. In general, a transition system which satisfies all instances of (BN) and (FA) is called *finitary*. The transition system induced by the SFP-domain $\mathcal{D}$ is finitary [Abr91b].

There are two reasons for considering finitary transitions systems. Semantically, finitary transitions systems are exactly those transition systems for which $\lesssim^F$ and $\lesssim^B$ coincide (see Lemma 22). Logically, the axioms (BN) and (FA) allow us to rewrite a formula in $\mathcal{L}^\pi_{\infty,\infty}$ as a conjunction of disjunctions of formulae in $\mathcal{L}_{\omega,\omega}$. This fact will be essential in the proof of our completeness resuts.

Lemma 21. *For each ϕ in $\mathcal{L}^\pi_{\infty,\infty}$ there exist formulae $\phi_i \in \mathcal{L}^\pi_{\omega,\infty}$, $i \in I$, such that $\mathcal{L}^\pi_{\infty,\infty} + (BN) + (FA) \vdash \phi =_\pi \bigwedge_I \phi_i$.*

Proof. See Lemma 5.17 of [Abr91b] and Lemma 17. □

An immediate consequence of the above lemma is the following characterization property. For a finitary transition system $\langle P, Act, \longrightarrow, \Uparrow\rangle$ and p, q in P,

$p \lesssim^F q$ if and only if $\forall \phi \in \mathcal{L}^\pi_{\infty,\infty}$: $p \models \phi \Rightarrow q \models \phi$.

By Theorem 12 it follows that for finitary transition systems, $\lesssim^F$ and $\lesssim^B$ coincide, while, by the duality Theorem 14, it follows that the order of $\mathcal{D}$, which is equivalent to the specialization order of $\mathcal{O}(\mathcal{D})$, coincides with the finitary preorder $\lesssim^F$. Therefore, in $\mathcal{D}$, $d_1 \leq d_2$ if and only if $d_1 \lesssim^B d_2$, that is, $\mathcal{D}$ is internally fully abstract with respect to partial bisimulation.

Finitary transition systems can be characterized in terms of partial bisimulation as follows.

Lemma 22. *For any transition system $T = \langle P, Act, \longrightarrow, \Uparrow \rangle$ the following conditions are equivalent:*

1. *T is finitary,*
2. *for all $p \in P$, p and $TS[\![p]\!]$ are equivalent in the bisimulation preorder $\lesssim^B$,*
3. *the finitary preorder $\lesssim^F$ coincides with bisimulation preorder $\lesssim^B$ in the transition system obtained as the disjoint union of T and $\mathcal{D}$.*

Proof. See Lemma 5.22 of [Abr91b]. □

In the last condition of the above lemma we need to consider the disjoint union of T and $\mathcal{D}$ because T alone may not have enough processes to prove the equivalence between $\lesssim^F$ and $\lesssim^B$.

To prove the completeness of the logic $\mathcal{L}^{\pi}_{\infty,\infty}$ for the class of finitary transition systems, consider its Lindenbaum algebra $\mathcal{LA}^{\pi}_{\infty,\infty}$ with as elements equivalence classes of formulae provable equivalent in $\mathcal{L}^{\pi}_{\infty,\infty} + (BN) + (FA)$. The logical axioms say that the poset $\mathcal{LA}^{\pi}_{\infty,\infty}$ is a completely distributive lattice. By Lemma 21 and with a proof similar to the proof of Lemma 18, it is not hard to see that $\mathcal{LA}^{\pi}_{\infty,\infty}$ enjoys universal properties.

Lemma 23. *The completely distributive lattice $\mathcal{LA}^{\pi}_{\infty,\infty}$ is free over the frame $\mathcal{LA}^{\pi}_{\omega,\infty}$.* □

By Theorem 3 it follows that the inclusion function

$$\iota\colon \mathcal{LA}^{\pi}_{\omega,\infty} \hookrightarrow \mathcal{LA}^{\pi}_{\infty,\infty}$$

is the free observation frame over $\mathcal{LA}^{\pi}_{\omega,\infty}$.

Lemma 24. *The assignment $[\phi] \mapsto [\![\phi]\!]^{\pi}_{\mathcal{D}}$ defines the unique order isomorphism $\gamma^{\star}\colon \mathcal{LA}^{\pi}_{\infty,\infty} \to \mathcal{Q}(\mathcal{D})$ such that $\gamma^{\star}([\phi]) = \gamma([\phi])$ for all $\phi \in \mathcal{L}^{\pi}_{\omega,\omega}$.*

Proof. Because $\mathcal{D}$ is an SFP-domain, if it is equipped with the Scott topology then it forms a sober space. Hence, by Theorem 7, the lattice of saturated sets $\mathcal{Q}(\mathcal{D})$ is the free completely distributive lattice over the frame of Scott open sets $\mathcal{O}(\mathcal{D})$, which, by Lemma 19, is order isomorphic to the Lindenbaum algebra $\mathcal{LA}^{\pi}_{\omega,\infty}$. But $\mathcal{LA}^{\pi}_{\infty,\infty}$ is the free completely distributive lattice over the frame $\mathcal{LA}^{\pi}_{\omega,\infty}$ (Lemma 23), and hence $\mathcal{Q}(\mathcal{D})$ is order isomorphic to $\mathcal{LA}^{\pi}_{\infty,\infty}$. The isomorphism is given by the unique extension $\gamma^{\star}$ of the function

$\gamma^+ : \mathcal{LA}^\pi_{\omega,\infty} \to \mathcal{O}(\mathcal{D})$ which can be characterized (using Lemma 17, the soundness Theorem 13, and the duality Theorem 19) by

$$\gamma^\star([\phi]) = \bigcap_I \gamma^+([\phi_i]) = \bigcap_I [\![\phi_i]\!]^\pi_{\mathcal{D}} = [\![\bigwedge_I \psi_i]\!]^\pi_{\mathcal{D}} = [\![\phi]\!]^\pi_{\mathcal{D}},$$

for all ϕ in $\mathcal{L}^\pi_{\infty,\infty}$. □

As before, soundness of the logical system associated with $\mathcal{L}^\pi_{\omega,\infty}$ including both the finitary schemes (BN) and (FA) follows from Theorem 13 and from the definition of finitary transition systems. In a similar way to the proof of the completeness Theorem 15, completeness follows from the above duality result.

Theorem 25. *(Completeness) Let $\mathcal{FT}$ be any class of finitary transition systems containing $\mathcal{D}$. For ϕ_1 and ϕ_2 in $\mathcal{L}^\pi_{\infty,\infty}$, $\mathcal{FT} \models \phi_1 \leq \phi_2$ if and only if $\mathcal{L}^\pi_{\infty,\infty} + (BN) + (FA) \vdash \phi_1 \leq \phi_2$.* □

7 Conclusion

The techniques we used in this paper are general and can be applied to every logic based on a topological interpretation. In particular we focussed on an infinitary extension of Abramsky's logic for transition systems. This logic is more expressive than the finitary one even for the class of finitary transition systems, and it can be used as a specification formalism. Our main motivation for the introduction of an infinitary domain logic as specification formalism is not to improve over the known specification tools but rather to analyse them by means of general and reusable mathematical notions of topology and domain theory (examples in this direction include a domain logic for Gamma [GH94] which was originally formulated as a transition assertion logic [EHJ93], and a domain logic for a shared-variable parallel language [Zha91] which was originally formulated by Brookes [Bro85]). This is part of Abramsky's general programme of connecting domain theory and operational notions of observability with denotational semantics and program logics.

In this paper we have not fully provided the infinitary domain logics for the specific languages of Abramsky for properties, typed terms, and morphisms. They are interpreted, respectively, as compact opens of SFP-domains, elements of SFP-domains, and morphisms between SFP-domains freely generated by a language of type expressions. It is surely worthwhile to work out these infinitary extensions in detail, but we leave it as future work.

The present paper does not deal with a formal comparison between Abramsky's logic and Hennessy-Milner logic for transition systems. Such a comparison can be found in [Abr91b], where $\mathcal{L}_{\infty,\infty}$ is proved equivalent to the infinitary Hennessy-Milner logic in the sense that a process of a transition system satisfies a formula of Abramsky's logic if and only if it satisfies the equivalent formula in the Hennessy-Milner logic. As a consequence, the complete distributive lattice induced by the (ordinary) interpretation of the infinitary Hennessy-Milner

logic coincides with the one induced by the infinitary Abramsky's logic. Thus the infinitary Hennessy-Milner logic together with a proof theory which one can get automatically by the equivalence above and extended with the axiom schemas (BN) and (FA) is isomorphic to the completely distributive lattice of saturated sets of the SFP-domain $\mathcal{D}$. From this we can conclude that the infinitary Hennessy-Milner logic extended with the axiom schemas (BN) and (FA) is sound and complete for any class of finitary transition systems containing $\mathcal{D}$.

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