Towards an infinitary logic of domains: Abramsky logic for transition systems

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Abstract

We give a new characterization of sober spaces in terms of their completely distributive lattice of saturated sets. This characterization is used to extend Abramsky's results about a domain logic for transition systems. The Lindenbaum algebra generated by the Abramsky finitary logic is a distributive lattice dual to an SFP-domain obtained as a solution of a recursive domain equation. We prove that the Lindenbaum algebra generated by the infinitary logic is a completely distributive lattice dual to the same SFP-domain. As a consequence soundness and completeness of the infinitary logic is obtained for a class of transition systems that is computational interesting.

1 Introduction

Complete partial orders were originally introduced as a mathematical structure to model computation [Sco70], in particular as domains for denotational semantics [SS71]. Successively, Scott's presentation of domains as information systems [Sco82] suggested a connection between denotational semantics and logics of programs. Based on the fundamental insight of Smyth [Smy83] that a topological space may be seen as a 'data type' with the open sets as 'observable predicates', and functions between topological spaces as 'computations', Abramsky [Abr87, Abr91a], Zhang [Zha91] and Vickers [Vic89] developed a propositional program logic from a denotational semantics.

Abramsky [Abr87, Abr91a] uses Stone duality to relate two views of SFP-domains (a special kind of complete partial orders): one in terms of logic theories and one in terms of semantic models. Abramsky's starting point is that for an algebraic cpo P, its compact elements completely determine P, whereas for a logic the Lindenbaum algebra provides a model from which the logic can be recovered.

If P is an SFP-domain, then the collection $\mathcal{KO}(P)$ of all Scott compact open subsets of P ordered by subset inclusion forms a distributive lattice. The distributive lattice $\mathcal{KO}(P)$ can be viewed as the Lindenbaum algebra of a logic.

Conversely, every logic such that its Lindenbaum algebra is a distributive lattice L gives rise to a spectral space by taking the collection of all prime filters of L as points together with the filter topology [Joh82]. Spectral spaces include SFP-domains when taken with the Scott topology.

Abramsky [Abr87] gives a duality for SFP-domains that can be built up in a modular way. He considers a number of basic constructors of domain theory, including lift, coalesced and separated sum, products, function space, Hoare, Smyth and Plotkin powerdomains, and recursion. Using the duality he shows that these constructors can be applied to Lindenbaum algebras dual to SFP-domains, and hence can be used to generate logics for constructors applied to SFP-domains. Abramsky's theory applies therefore to all SFP-domains freely generated by the constructors.

Although mathematically very attractive, the logics of compact opens considered by Abramsky are weak in expressive power, and inadequate as a general specification formalism according to [Abr87]. What is needed is a language, with an accompanying semantic framework, which permits to go beyond compact open sets. In particular, there is the need for an infinitary propositional logic with infinite disjunctions and infinite conjunctions.

Since the spaces considered by Abramsky are spectral, the introduction of infinite disjunctions does not require a major adjustment of the semantic framework: we can consider the whole frame of open sets which is free over the distributive lattice of compact opens [Joh82].

The addition of infinite conjunctions is more difficult because it requires new mathematical tools which we present in this paper. We use the theory of observation frames [BJK95] to derive a new characterization of sober spaces in terms of the completely distributive lattice of saturated sets. This result allows us to freely extend the finitary logic of compact opens to the infinitary logic of saturated sets. The extension is conservative in the sense that the topological space represented by a finitary logic coincides with the one represented by its infinitary extension. The techniques involved are general and can be applied to every logic based on a topological interpretation.

As an application we treat Abramsky's domain logic for labeled transition systems with divergence [Abr91b]. Abramsky's domain logic for transition systems is equivalent to the Hennessy-Milner logic in the infinitary case, and hence it characterizes bisimulation for every transition system. However in the finitary case it is more satisfactory than the Hennessy-Milner logic in the sense that it characterizes a finitary preorder (the finitary observable part of bisimulation) for every transition system. Abramsky's infinitary logic can be used to characterize the class of transition systems for which the bisimulation preorders are algebraic, in the sense that they coincide with the finitary preorders. These transition systems are called finitary and satisfy two axiom schemes: one about bounded nondeterminism and another one about finite approximation.

We prove soundness and completeness of the infinitary logic for the class of all finitary transition systems. The same completeness result holds also for the infinitary Hennessy-Milner logic because the latter is equivalent to Abramsky's infinitary logic. On the way to proving our completeness result, we also show soundness and completeness of Abramsky's logic with infinite disjunctions for the class of compactly branching transition systems.

The paper is based on [BK97] and it is organized as follows. In Section 2 we give some basic definitions and facts about distributive lattices. All material presented in this section is standard, except for the construction of the free completely distributive lattice over a set. Next we give in Section 3 a classification of topological spaces in terms of their completely distributive lattice of saturated sets. We consider spectral spaces and sober spaces. Using a duality between $\mathcal{T}_0$ spaces and observation frames, we characterize (1) spectral spaces as those spaces for which their completely distributive lattice of saturated sets is free over the distributive lattice of compact opens, and (2) sober spaces as those spaces for which their completely distributive lattice of saturated sets is free over the frame of opens. In Section 4 we discuss how these two characterizations allow for an infinitary logic of domains which extend the finitary framework of Abramsky [Abr87].

A concrete example of infinitary logic of domains involving the Plotkin powerdomain construction is treated in the subsequent sections. In Section 5 we introduce Abramsky's infinitary domain logic for labeled transition systems, and prove the completeness of its finitary restriction. Then, in Section 6 we prove the completeness of the restricted logic with arbitrary disjunctions and finite conjunctions for the class of compactly branching transition systems. Finally, in Section 7 we prove the completeness of the entire infinitary logic for the class of finitary transition systems.

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2 Completely distributive lattices

In this section we give some basic definitions and facts about distributive lattices, and show how to construct frames from distributive lattices, and completely distributive lattices from frames. These constructions will be used in the next section to characterize classes of topological spaces in terms of free properties satisfied by their completely distributive lattice of saturated sets.

A subset S of a poset P is *lower closed* if $x \in S$ and $y \leq x$ implies $y \in S$. Dually, S is *upper closed* if $x \in S$ and $x \leq y$ implies $y \in S$. The set S is said to be *directed* if for each pair of elements x and y in S there exists $z \in S$ such that $x \leq z$ and $y \leq z$. Below we write $\bigvee S$ and $x \vee y$ for the join of an arbitrary subset S of P and the binary join of two elements in P, respectively, if they exist. Dually, we denote by $\bigwedge S$ and $x \wedge y$ the meet of an arbitrary subset S of P and the binary meet of two elements in P, respectively.

A lattice L is called *distributive* if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

for all a, b and c in L. The above equation holds for a lattice if and only if so does its dual [Sc890], where we substitute meets for joins and joins for meets. The class of all distributive lattices together with functions preserving both finite meets and finite joins defines a category, denoted by **DLat**.

If the lattice L has join for arbitrary subsets, and not just finite ones, and it satisfies the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \mid s \in S\}$$

for all $a \in L$ and all subsets $S \subseteq L$ then it is called a *frame*. Frames with functions preserving arbitrary joins and finite meets form a category called **Frm**. There is an obvious forgetful functor from **Frm** to **DLat**.

Proposition 2.1 *For each distributive lattice L, the poset $Idl(L)$ of all directed and lower closed subsets of L ordered by subset inclusion forms a frame. Moreover, the assignment $L \mapsto Idl(L)$ can be extended to a functor from* **DLat** *to* **Frm** *which is left adjoint to the forgetful functor* **Frm** $\to$ **DLat**.

Proof: See Corollary II.2.11 in [Joh82]. □

A complete lattice L is *completely distributive* if, for all sets $\mathcal{A}$ of subsets of L,

$$\bigwedge \{\bigvee S \mid S \in \mathcal{A}\} = \bigvee \{\bigwedge f(\mathcal{A}) \mid f \in \Phi(\mathcal{A})\},$$

where $f(\mathcal{A})$ denotes the set $\{f(S) \mid S \in \mathcal{A}\}$ and $\Phi(\mathcal{A})$ is the set of all functions $f\colon \mathcal{A} \to \bigcup \mathcal{A}$ such that $f(S) \in S$ for all $S \in \mathcal{A}$. The above equation holds for a lattice if and only if so does its dual [Ran52], where we substitute meets for joins and joins for meets. Completely distributive lattices with functions preserving both arbitrary meets and arbitrary joins form a category, denoted by **CDL**. Clearly every completely distributive lattice is a frame.

Next we construct the free completely distributive lattice over a set. The construction we present is similar to the free frame construction and differs only slightly from the construction presented (without proof) in [Mar79]. For a set X, let $CDL(X)$ denote the collection of all lower closed subsets of the poset $(\mathcal{P}(X), \supseteq)$ ordered by subset inclusion. Since $CDL(X)$ is closed under arbitrary unions and arbitrary intersections, it is a complete sub-lattice of $\mathcal{P}(\mathcal{P}(X))$. Hence $CDL(X)$ is a completely distributive lattice.

The set X can be mapped into $CDL(X)$ by the function $\theta_X\colon X \to CDL(X)$ defined by

$$\theta_X(x) \quad = \quad \{S \subseteq X \mid x \in S\},$$

for every $x \in X$. The above construction is universal.

Theorem 2.2 *Let X be a set and L be a completely distributive lattice. For any function $f: X \to L$ there exists a unique morphism $f^\dagger: CDL(X) \to L$ in* **CDL** *such that $f^\dagger \circ \theta_X = f$.*

Proof: For every element q in $CDL(X)$, it holds

$$q \quad = \quad \bigcup\{\bigcap\{\theta_X(x) \mid x \in S\} \mid S \in q\}. \tag{1}$$

Since $f^\dagger: CDL(X) \to L$ preserves arbitrary joins and arbitrary meets, and $f^\dagger \circ \theta_X = f$, its only possible definition is given, for $J \in CDL(X)$, by

$$f^\dagger(q) = \bigvee\{\bigwedge\{f(x) \mid x \in S\} \mid S \in q\}.$$

From the form of the above definition it follows that $f^\dagger$ preserves arbitrary joins. So it remains to prove that $f^\dagger$ preserves all meets. Let $q_i \in CDL(X)$ for all i in an arbitrary set I, and let $h: \mathcal{P}(X) \to L$ be the function mapping every subset S of X to $\bigcap\{f(x) \mid x \in S\}$. It is not hard to see that h preserves arbitrary meets. Moreover $f^\dagger(q) = \bigvee\{h(S) \mid S \in q\}$. We have

$$\begin{aligned}
\bigwedge\{f^\dagger(q_i) \mid i \in I\} &= \bigwedge\{\bigvee\{h(S) \mid S \in q_i\} \mid i \in I\} \\
&= \bigvee\{\bigwedge\{h(g(i)) \mid i \in I\} \mid g \in \Phi(I)\} \quad \text{[complete distributivity]} \\
&= \bigvee\{h(\bigwedge\{g(i) \mid i \in I\}) \mid g \in \Phi(I)\} \quad [h \text{ preserves meets}] \\
&= \bigvee\{h(S) \mid S \in \bigcap\{q_i \mid i \in I\}\} \quad [\text{all } q_i\text{'s are lower sets}] \\
&= f^\dagger(\bigcap\{q_i \mid i \in I\}),
\end{aligned}$$

where $\Phi(I)$ is the set of all functions $g: I \to \bigcup_I q_i$ such that $g(i) \in q_i$. □

The above theorem implies that the assignment $X \mapsto CDL(X)$ can be extended to a functor CDL: **Set** $\to$ **CDL** which is a left adjoint to the forgetful functor **CDL** $\to$ **Set**. It follows that the category **CDL** is algebraic, because **CDL** is clearly equationally presentable (i.e. its objects can be described by a proper class of operations and equations) [Man76, Chapter 1]. Also the category **Frm** is algebraic [Joh82, Theorem *II*.1.2]. Hence the forgetful functor **CDL** $\to$ **Frm** has a left adjoint denoted by $\overline{(\cdot)}$: **Frm** $\to$ **CDL**. Next we give a more direct proof of this fact.

For a frame F define $\widehat{F}$ to be the set $\{\widehat{x} \mid x \in F\}$, and let $\equiv_F$ be the least congruence (with respect to arbitrary meets and arbitrary joins) on $CDL(\widehat{F})$ such that

$$\bigwedge_{x\in S} \widehat{x} \quad \equiv_F \quad \widehat{\bigwedge_{x \in S} x} \qquad \text{for every finite subset } S \text{ of } F \tag{2}$$

$$\bigvee_{x\in S} \widehat{x} \quad \equiv_F \quad \widehat{\bigvee_{x \in S} x} \qquad \text{for every subset } S \text{ of } F. \tag{3}$$

Define $\overline{F} = CDL(\widehat{F})/\equiv_F$. Because $\equiv_F$ is a congruence, we have that $\overline{F}$ is a completely distributive lattice. Finally, define $\zeta_F: F \to \overline{F}$, for each $x \in F$, as follows

$$\zeta_F(x) \quad = \quad [\widehat{x}]_F\,,$$

where $[\widehat{x}]_F$ denotes the set of elements of $CDL(\widehat{F})$ equivalent to $\widehat{x}$ under $\equiv_F$. By the equivalences (2) and (3) above, it follows that ζ_F is a frame morphism.

Lemma 2.3 *For every frame F, the completely distributive lattice $\overline{F}$ is order generated by the image of F under $\zeta_F: F \to \overline{F}$.*

Proof: It is enough to prove that each element of $\overline{F}$ is the meet of elements in $\zeta_F(F)$. Let $[q]_F \in \overline{F}$. By Equation (1) and the dual of the complete distributive law (which holds for every completely distributive lattice) we obtain that in $CDL(\widehat{F})$,

$$q \quad = \quad \bigwedge_I \bigvee_{J_i} \widehat{x_{i,j}}$$

for some sets I and J_i, and elements $x_{i,j} \in F$. Because $\equiv_F$ is a congruence we obtain

$$[q] \quad = \quad \bigwedge_I [\bigvee_{J_i} \widehat{x_{i,j}}]_F \,.$$

For each $i \in I$, let $x_i = \bigvee_{J_i} x_{i,j}$. By definition of $\equiv_F$, $\widehat{x_i} \equiv_F \bigvee_{J_i} \widehat{x_{i,j}}$. Thus $\zeta_F(x_i) = [\bigvee_{J_i} \widehat{x_{i,j}}]_F$, from which it follows that $[q] = \bigwedge_I \zeta_F(x_i)$. □

We can use the above lemma to prove the following.

Theorem 2.4 *The assignment $F \mapsto \overline{F}$ can be extended to a functor from* **Frm** *to* **CDL** *which is a left adjoint to the forgetful functor* **CDL** $\to$ **Frm**. *The unit of the adjunction is given by the function $\zeta_F : F \to \overline{F}$.*

Proof: Let L be a completely distributive lattice, and let $f : F \to L$ be a frame morphism. We need to find a unique morphism $h : \overline{F} \to L$ in **CDL** such that $f \circ \zeta_F = h$. Because $\overline{F}$ is order generated by ζ_F, and h must preserve arbitrary meets, the only possible definition for h is

$$h(q) \quad = \quad \bigwedge \{ f(x) \mid x \in F \text{ and } q \leq \zeta_F(x) \} \,.$$

Clearly $h(\zeta_F(x)) = f(x)$, and h preserves arbitrary meets. Preservation of arbitrary joins can be proved using the complete distributive law. □

It should be remarked here that we do not know of any direct construction adding the "missing" codirected meets to a frame while preserving both the existing finite meets and arbitrary joins. The intuitively appealing filter completion of a frame does not work as is shown in [Bon97, Chapter 9].

3 Completely distributive lattices and topological spaces

In this section we give a classification of topological spaces in terms of their completely distributive lattice of saturated sets. Our purpose is to derive a new characterization of sober spaces which will be the key mathematical ingredient of the next sections, where it will be used to prove the completeness of an infinitary propositional theory based on an existing completeness result of its finitary restriction.

For a frame F let $\zeta_F : F \to \overline{F}$ be the unit of the adjunction between **Frm** and **CDL**. We have seen that ζ_F is a frame morphism and that every element of the completely distributive lattice $\overline{F}$ is the meet of elements in $\zeta_F(F)$, that is, $\overline{F}$ is order generated by $\zeta_F(F)$. In general we call a map with these properties an observation frame.

Definition 3.1 *An* observation frame *is a frame morphism $\alpha : F \to L$ between a frame F and a completely distributive lattice L such that, for every $q \in L$,*

$$q \quad = \quad \bigwedge \{ \alpha(x) \mid x \in F \textit{ and } q \leq \alpha(x) \}.$$

Observation frames can be organized into a category, denoted by **OFrm**, with arrows defined as follows. A morphism between two observation frames $\alpha : F \to L$ and $\beta : G \to H$ is a pair $\langle f, g \rangle$ consisting of a frame morphism $f : F \to G$ and a complete distributive lattice morphism $g : L \to H$ such that $g \circ \alpha = \beta \circ f$ [BJK95, Bon97].

There is a functor Dom: **OFrm** $\to$ **Frm** mapping an observation frame $\alpha : F \to L$ to the frame $Dom(\alpha) = F$ and a morphism $\langle f, g \rangle$ in **OFrm** to the frame morphism $Dom(\langle f, g \rangle) = f$.

Theorem 3.2 *The functor Dom:* **OFrm** $\to$ **Frm** *has a left adjoint.*

Proof: Let F be a frame and consider the observation frame $\zeta_F: F \to \overline{F}$ defined as the unit of the adjunction given in Theorem 2.4. The identity function $id_F: F \to Dom(\zeta_F)$ is clearly a frame morphism. Moreover for every other observation frame $\beta: G \to H$ and frame morphism $f: F \to Dom(\beta)$ by Theorem 2.4 there exists a unique morphism $g: \overline{F} \to H$ such that $g \circ \zeta_F = \beta \circ f$. Hence $\langle f, g\rangle$ is the unique morphism in **OFrm** from ζ_F to β such that $Dom(\langle f, g\rangle) \circ id_F = f$. □

Observation frames were introduced in [BJK95] in order to represent abstractly topological spaces: if X is a topological space, the inclusion $\mathcal{O}(X) \hookrightarrow \mathcal{Q}(X)$ mapping the frame of open sets into the completely distributive lattice of the saturated subsets of X forms an observation frame. We denote it by $\Omega(X)$. Moreover, if $f: X \to Y$ is a continuous function between spaces X and Y (i.e. a map in the category of topological spaces **Sp**) then

$$\Omega(f) = \langle f^{-1}: \mathcal{O}(Y) \to \mathcal{O}(X), f^{-1}: \mathcal{Q}(Y) \to \mathcal{Q}(X)\rangle$$

is a morphism in **OFrm** between $\Omega(Y)$ and $\Omega(X)$. Thus we have a functor $\Omega: \mathbf{Sp} \to \mathbf{OFrm}^{op}$.

Next we show that Ω has a right adjoint. For an observation frame $\alpha: F \to L$, a filter $\mathcal{F}$ of F is said to be an *M-filter* if, for all $x \in F$,

$$\bigwedge \alpha(\mathcal{F}) \leq \alpha(x) \Rightarrow x \in \mathcal{F}\,.$$

We denote by $CPMF(\alpha)$ the set of all completely prime M-filters of an observation frame α, and by $CPF(F)$ the set of all completely prime filters of a frame F. Clearly, for $\alpha: F \to L$, $CPMF(\alpha) \subseteq CPF(F)$.

Lemma 3.3 *The collection of all completely prime filters of a frame F coincides with the collection of all completely prime M-filters of the free observation frame $\zeta_F: F \to \overline{F}$.*

Proof: We need to prove that each completely prime filter of F is an M-filter of $\zeta_F: F \to \overline{F}$.

As a consequence of Theorem 3.2 and Theorem 2.4, the assignment $f \mapsto \langle f, \overline{f}\rangle$ is an isomorphism, natural in both F and α, between

$$\mathbf{Frm}(F, Dom(\alpha)) \cong \mathbf{OFrm}(\zeta_F, \alpha)\,. \tag{4}$$

Let $2 = \{\bot, \top\}$ be the two point completely distributive lattice (with $\bot \leq \top$) and $id_2: 2 \to 2$ the identity function on 2. Clearly id_2 is an observation frame. By the above isomorphism $f \in \mathbf{Frm}(F, 2)$ if and only if there exists a morphism $g: \overline{F} \to 2$ between completely distributive lattices such that $\langle f, g\rangle \in \mathbf{OFrm}(\zeta_F, id_2)$.

Recall that completely prime filters of a frame F can be characterized as sets of the form $f^{-1}(\top)$ for $f \in \mathbf{Frm}(F, 2)$ [Vic89, Proposition 5.4.7], and, similarly, completely prime M-filters of an observation frame $\alpha: F \to L$ are exactly sets of the form $f^{-1}(\top)$ for $\langle f, g\rangle \in \mathbf{OFrm}(\alpha, id_2)$ [BJK95, Lemma 3.16]. Hence $CPMF(\zeta_F)$ coincides with $CPF(F)$. □

For an observation frame $\alpha: F \to L$ we denote by $OPt(\alpha)$ the topological space given by the set $CPMF(\alpha)$ of all completely prime M-filters of α together with a topology with open sets defined, for every $x \in F$, by

$$\{\mathcal{F} \in CPMF(\alpha) \mid x \in \mathcal{F}\}.$$

An observation frame $\alpha: F \to L$ is called *spatial* if for each $x, y \in F$ whenever $x \not\leq y$ then there exists $\mathcal{F} \in CPMF(\alpha)$ such that $x \in \mathcal{F}$ but $y \notin \mathcal{F}$.

Theorem 3.4 *The assignment $\alpha \mapsto OPt(\alpha)$, where $\alpha: F \to L$ is an observation frame, can be extended to a functor from $\mathbf{OFrm}^{op}$ to* **Sp** *which is right adjoint of Ω. The unit of the adjunction is given by the assignment*

$$x \mapsto \{o \in \mathcal{O}(X) \mid x \in o\}\,.$$

Furthermore, the adjunction restricts to an equivalence between the full subcategories $\mathbf{Sp}_0$ of $\mathcal{T}_0$ spaces and **SOFrm** *of spatial observation frames.*

Proof: See Theorem 3.23 and Corollary 3.30 in [BJK95]. □

3.1 Sober spaces

Traditionally, topological spaces can be represented abstractly by considering the frame of open sets. There is a functor $\mathcal{O}(-)\colon \mathbf{Sp} \to \mathbf{Frm}^{op}$ which maps every topological space to its lattice of open sets and every continuous function to its inverse restricted to the open sets. Conversely, given a frame F, we can construct a topological space $FPt(F)$ by taking the set $CPF(F)$ of all completely prime filters of F, together with a topology with open sets defined, for every $x \in F$, by

$$\{\mathcal{F} \in CPF(F) \mid x \in \mathcal{F}\}.$$

The space $FPt(F)$ is sober, where a space X is said to be *sober* if the assignment

$$x \mapsto \{o \in \mathcal{O}(X) \mid x \in o\}$$

defines an isomorphism between X and $CPF(\mathcal{O}(X))$. Also, a frame F is called *spatial* if for each x and y in F, whenever $x \not\leq y$ then there exists $\mathcal{F} \in CPF(F)$ such that $x \in \mathcal{F}$ but $y \notin \mathcal{F}$.

Proposition 3.5 *The assignment $F \mapsto FPt(F)$ defines a functor $\mathbf{Frm}^{op} \to \mathbf{Sp}$ which is a right adjoint of $\mathcal{O}(-)\colon \mathbf{Sp} \to \mathbf{Frm}^{op}$. Furthermore we have that*

1. *the adjunction restricts to a duality between the full subcategories $\mathbf{Sob}$ of sober spaces and $\mathbf{SFrm}$ of spatial frames;*

2. *the inclusion $\mathbf{Sob} \hookrightarrow \mathbf{Sp}_0$ has left adjoint $FPt(\mathcal{O}(-))$.*

Proof: See Theorem *II*.1.4 and Corollary *II*.1.7 in [Joh82]. $\square$

By Lemma 3.3, a frame F is spatial if and only if the observation frame $\zeta_F\colon F \to \overline{F}$ is spatial. Hence the adjunction of Theorem 3.2 restricts to an adjunction between the category of spatial frames $\mathbf{SFrm}$ and the category of spatial observation frames $\mathbf{SOFrm}$. Since adjoints are defined uniquely (up to natural isomorphisms), the above implies that commutativity of the rounded squares below.

$$\begin{tikzcd}
\mathbf{Sp}_0 \arrow[rr, bend right, "\Omega(-)"'] \arrow[rr, phantom, "\simeq"] \arrow[d, bend right, "FPt(\mathcal{O}(-))"'] \arrow[d, phantom, "\dashv"] & & \mathbf{SOFrm}^{op} \arrow[ll, bend right, "OPt(-)"'] \arrow[d, bend right, "Dom(-)^{op}"'] \arrow[d, phantom, "\dashv"] \\
\mathbf{Sob} \arrow[u, hook, bend right, "i"'] \arrow[rr, bend right, "\mathcal{O}(-)"'] \arrow[rr, phantom, "\simeq"] & & \mathbf{SFrm}^{op} \arrow[ll, bend right, "FPt(-)"'] \arrow[u, bend right, "\zeta^{op}_{(-)}"']
\end{tikzcd}$$

The functor $Dom\colon \mathbf{OFrm} \to \mathbf{Frm}$ can therefore be considered as the pointless sobrification of an abstract topological space. Now we use the above results to derive a new characterization of sober spaces.

Theorem 3.6 *A $\mathcal{T}_0$ space X is sober if and only if the completely distributive lattice of saturated sets $\mathcal{Q}(X)$ is free over the frame of open sets $\mathcal{O}(X)$.*

Proof: Assume X is a sober space. By the commutativity of the above diagram it follows that the observation frames $\Omega(X)\colon \mathcal{O}(X) \to \mathcal{Q}(X)$ and $\zeta_{\mathcal{O}(X)}\colon \mathcal{O}(X) \to \overline{\mathcal{O}(X)}$ are isomorphic in $\mathbf{OFrm}$. Hence $\overline{\mathcal{O}(X)}$ is isomorphic to $\mathcal{Q}(X)$ in $\mathbf{CDL}$. But $\overline{\mathcal{O}(X)}$ is the free completely distributive lattice over the frame $\mathcal{O}(X)$, by Theorem 2.4.

For the converse, assume X is a $\mathcal{T}_0$ space and $\mathcal{Q}(X)$ is the free completely distributive lattice over the frame $\mathcal{O}(X)$. Then the set of all completely prime M-filters of $\Omega(X)$ coincides with the set of all completely prime M-filters of $\zeta_{\mathcal{O}(X)}$, which, by Lemma 3.3, coincides with the set of all completely prime filters of $\mathcal{O}(X)$. Since X is a $\mathcal{T}_0$ space, the assignment

$$x \mapsto \{o \in \mathcal{O}(X) \mid x \in o\}$$

is an isomorphism between X and $\mathit{CPMF}(\Omega(X))$. But

$$\mathit{CPF}(\mathcal{O}(X)) = \mathit{CPMF}(\Omega(X)),$$

hence X is a sober space. □

3.2 Spectral spaces

A $\mathcal{T}_0$ space X is *spectral* if the set $\mathcal{KO}(X)$ of compact open subsets of X forms a basis for the topology of X, and it is closed under finite intersections. Since basic opens are closed under finite unions, $\mathcal{KO}(X)$ is a distributive lattice. The class of all spectral spaces together with continuous functions preserving compact opens under inverse image defines a category, denoted by **Spec**. If $f\colon X \to Y$ is a morphism in **Spec** then

$$\mathcal{KO}(f) \quad = \quad f^{-1}\colon \mathcal{KO}(Y) \to \mathcal{KO}(X)$$

is a lattice morphism. Thus we have a functor $\mathcal{KO}(-)\colon \mathbf{Spec} \to \mathbf{DLat}^{op}$.

Conversely, for a distributive lattice L, let $\mathit{Spec}(L)$ be the topological space of prime filters over L with topology generated by the sets

$$\{\mathcal{F} \in \mathit{Spec}(L) \mid a \in \mathcal{F}\},$$

for $a \in L$. The above sets are compact in the space $\mathit{Spec}(L)$ and closed under finite unions and finite intersections.

Proposition 3.7 *For every distributive lattice L, $\mathit{Spec}(L)$ is isomorphic to $\mathit{FPt}(\mathit{Idl}(L))$ in* **Sp**. *Furthermore, the duality of Proposition 3.5 restricts to a duality between the categories* **Spec** *of spectral spaces and* **DLat** *of distributive lattices.*

Proof: See Corollary *II*.3.3 in [Joh82]. □

Combining the above result with Proposition 2.1 and Theorem 3.6 we obtain the following.

Theorem 3.8 *For a $\mathcal{T}_0$ space X the following are equivalents:*

1. *X is spectral;*
2. *the frame of open sets $\mathcal{O}(X)$ is free over the distributive lattice of compact opens $\mathcal{KO}(X)$;*
3. *the completely distributive lattice of saturated sets $\mathcal{Q}(X)$ is free over the distributive lattice of compact opens $\mathcal{KO}(X)$.* □

4 Domain theory in logical form

In this section we briefly discuss Abramsky's framework [Abr87, Abr91a] for connecting denotational semantics and program logic, and explain how the results of the previous section can be used to extend it.

Abramsky's starting point is that a lattice can be thought of as the Lindenbaum algebra $\mathcal{LA}$ of a propositional theory $\mathcal{L} = (L, \leq)$, where L is a set of formulae and $\leq$ is the relation of logical entailment between formulae. The elements of $\mathcal{LA}$ are equivalence classes of formulae provably equivalent in $\mathcal{L}$, meets are logical conjunctions, and joins are logical disjunctions.

A model of $\mathcal{L}$ is a set X together with a satisfaction relation $\models \subseteq X \times \mathcal{L}$ that is consistent with the logic of $\mathcal{L}$, i.e. such that an element of X satisfies a disjunction of formulae if and only if it satisfies at least one of them, and it satisfies a conjunction of formulae if and only if it satisfies all of them. This interpretation is automatically sound, in the sense that whenever $\phi \leq \psi$ in $\mathcal{L}$ then $x \models \phi$ implies $x \models \psi$. Conversely, the interpretation is complete if whenever $x \models \phi$ implies $x \models \psi$ for every $x \in X$ then $\phi \leq \psi$ in $\mathcal{L}$. If $(X, \models)$ is a sound and complete model of $\mathcal{L}$ then the Lindenbaum algebra $\mathcal{LA}$ must be distributive. This follows because the set of all $[\![\phi]\!]$ for $\phi \in L$ ordered by subset inclusion is a sub-lattice of $\mathcal{P}(X)$ and hence is distributive, where $[\![\phi]\!] = \{x \in X \mid x \models \phi\}$. By Proposition 3.7, if $\mathcal{LA}$ is distributive then there exists a sound and complete model for $\mathcal{L}$, namely the set of prime filters of $\mathcal{LA}$ together with the satisfaction relation

$$\mathcal{F} \models \phi \quad \text{if and only if} \quad [\phi] \in \mathcal{F}$$

where $\mathcal{F}$ is a prime filter of $\mathcal{LA}$, ϕ a formula in L, and $[\phi] \in \mathcal{LA}$ is the equivalence class of formulae logically equivalent to ϕ.

Abramsky considers a typed language together with a denotational interpretation which maps each type σ of the language to an SFP-domain $\mathcal{D}(\sigma)$. The language has several type constructors which are interpreted denotationally as the standard domain constructors, such as products, co-products, function space and powerdomains. Since $\mathcal{D}(\sigma)$ is an SFP-domain, the set of its Scott compact open subsets ordered by subset inclusion forms a distributive lattice, that is, $\mathcal{D}(\sigma)$ taken with the Scott topology is a spectral space [Plo81a, Chapter 8, Theorem 6].

A second logical interpretation associates to each type σ of the language a propositional theory $\mathcal{L}(\sigma) = (L(\sigma), \leq_\sigma)$. Each theory has axioms and rules which enforce a distributive lattice structure with finite meets and finite joins. Moreover, for each type constructor there is a corresponding constructor between propositional theories.

The logical interpretation and the denotational interpretation are connected as follows. For any type σ Abramsky defines a function

$$[\![\cdot]\!]_\sigma : L(\sigma) \to \mathcal{KO}(\mathcal{D}(\sigma))$$

which interprets formulae of $\mathcal{L}(\sigma)$ as compact open sets in the Scott topology of the SFP-domain $\mathcal{D}(\sigma)$. This function induces a satisfaction relation

$$\models_\sigma \subseteq \mathcal{D}(\sigma) \times L(\sigma)$$

for each $d \in \mathcal{D}(\sigma)$ and $\phi \in L(\sigma)$ by

$$d \models_\sigma \phi \quad \text{if and only if} \quad d \in [\![\phi]\!]_\sigma .$$

For each type of the language Abramsky proved that the model $(\mathcal{D}(\sigma), \models_\sigma)$ is sound and complete. This result is obtained for each type of the language in a uniform way via a number of steps including the following:

1. *Soundness.* Axioms and rules in $\mathcal{L}(\sigma)$ translate via $[\![\cdot]\!]_\sigma$ in valid statements about Scott compact opens of $\mathcal{D}(\sigma)$.

2. *Normal form.* Using the axioms and the rules each formula in $L(\sigma)$ is proved equivalent to a disjunction of formulae which are join-primes in the Lindenbaum algebra of $\mathcal{L}(\sigma)$. Here an element a of a lattice L is said to be *join-prime* if whenever $a \leq \bigvee S$ for some finite subset S of L then $a \leq b$ for some $b \in S$. By the soundness above, it follows that $[\![\cdot]\!]_\sigma$ restricts and corestricts to a map $[\![\cdot]\!]^0_\sigma$ from formulae that are join-primes in the Lindenbaum algebra of $\mathcal{L}(\sigma)$ to join-primes Scott compact opens of $\mathcal{D}(\sigma)$.

3. *Prime completeness.* The function $[\![\cdot]\!]^0_\sigma$ is proved order-reflecting.

4. *Prime definability.* The function $[\![\cdot]\!]^0_\sigma$ is proved surjective.

From the above results it follows that $(\mathcal{D}(\sigma), \models_\sigma)$ is sound and complete, and that $[\![\cdot]\!]_\sigma$ is an order pre-isomorphism [Abr91a] (see also [AJ94, Chapter 7]).

As a consequence of the Abramsky theory, an element of an SFP-domain can be considered equivalent to the set of all properties satisfied by that element, which therefore gives a logical characterization of it. Even more, the order of the SFP-domain can be characterized in terms of the properties satisfied by the elements, that is, one element is smaller or equal to a second element if and only if every property satisfied by the first element is also satisfied by the second one.

4.1 Towards an infinitary logic of domains

It is important to stress here that the propositional theories used by Abramsky for the logical interpretation of his type language are finite. They describe the logics of compact open sets which are mathematically very attractive because they are decidable and they represent the logics of observable properties [Abr87]. However they have weak expressive power and cannot specify typical safety and liveness properties of interest in computer science. In the next sections we will give some examples of properties that cannot be specified by Scott compact opens.

What is needed are propositional theories which allow for infinitary joins and infinitary meets. Next we informally discuss how infinitary propositional theories can be used for characterizing domains without major adjustments to Abramsky's framework. This is a consequence of the results of the Section 2 and Section 3. For each type σ of Abramsky's language we can proceed as follows:

1. *Definition.* This is the most 'creative' part of the 'enterprise'. We have to define a new logical interpretation $\mathcal{L}(\sigma)_{\infty,\infty}$ which allows for infinite joins and infinite meets, and an accompanying semantic function mapping formulae of the theory to saturated sets of $\mathcal{D}(\sigma)$.

2. *Coherence.* The restricted theory with finite joins and finite meets must coincide with Abramsky original theory $\mathcal{L}(\sigma)$, and the semantic interpretation of a formula ϕ in $\mathcal{L}(\sigma)$ is the original interpretation $[\![\phi]\!]_\sigma$.

3. *Soundness.* We have to prove that axioms and rules of $\mathcal{L}(\sigma)_{\infty,\infty}$ translate via the semantic function to valid statements about saturated sets of $\mathcal{D}(\sigma)$. It follows that meets are interpreted as conjunctions and joins as disjunctions.

4. *Conjunctive normal form.* Let $\mathcal{L}(\sigma)_{\omega,\infty}$ be the restricted theory with infinite joins and finite meets. If we can prove in $\mathcal{L}(\sigma)_{\infty,\infty}$ that each formula is equivalent to an (infinite) meet of formulae in $\mathcal{L}(\sigma)_{\omega,\infty}$, then it follows that the Lindenbaum algebra of $\mathcal{L}(\sigma)_{\infty,\infty}$ is the free completely distributive lattice over the frame induced by $\mathcal{L}(\sigma)_{\omega,\infty}$.

5. *Disjunctive normal form.* If we can prove in $\mathcal{L}(\sigma)_{\omega,\infty}$ that each formula is equivalent to an (infinite) join of formulae in $\mathcal{L}(\sigma)$, then it follows that the Lindenbaum algebra of $\mathcal{L}(\sigma)_{\omega,\infty}$ is the free frame over the distributive lattice induced by $\mathcal{L}(\sigma)$. Furthermore, by 'soundness' and 'coherence' the above implies that formulae in $\mathcal{L}(\sigma)_{\omega,\infty}$ are interpreted as Scott open subsets of $\mathcal{D}(\sigma)$.

6. *Isomorphism, I.* Since SFP-domains are spectral spaces when taken with the Scott topology, by Theorem 3.8 the lattice $\mathcal{O}(\mathcal{D}(\sigma))$ of Scott open subsets of $\mathcal{D}(\sigma)$ is the free frame over the distributive lattice $\mathcal{KO}(\mathcal{D}(\sigma))$ of Scott compact open subsets of $\mathcal{D}(\sigma)$. Since $[\![\cdot]\!]_\sigma$ is a pre-isomorphism between formulae in $\mathcal{L}(\sigma)$ and Scott compact opens of $\mathcal{D}(\sigma)$, the restriction of the semantic function to formulae in $\mathcal{L}(\sigma)_{\omega,\infty}$ and its corestriction to Scott open sets of $\mathcal{D}(\sigma)$ is also a pre-isomorphism.

7. *Isomorphism, II.* Since spectral spaces are sober spaces, by Theorem 3.6 the lattice $\mathcal{Q}(\mathcal{D}(\sigma))$ of saturated subsets of $\mathcal{D}(\sigma)$ (with respect to the Scott topology of $\mathcal{D}(\sigma)$) is the free completely distributive lattice over the frame $\mathcal{O}(\mathcal{D}(\sigma))$ of Scott open subsets of $\mathcal{D}(\sigma)$. By the above pre-isomorphism it follows that the semantic function is also a pre-isomorphism between formulae in $\mathcal{L}(\sigma)_{\infty,\infty}$ and upper closed sets of $\mathcal{D}(\sigma)$.

From the above results it follows that we can define a satisfaction relation $\models_\sigma$ such that $(\mathcal{D}(\sigma), \models_\sigma)$ is a sound and complete model for the theory $\mathcal{L}(\sigma)_{\infty,\infty}$.

5 Domain logic for transition systems

As an application of the techniques discussed above, we treat Abramsky's domain logic for labeled transition systems with divergence [Abr91b].

5.1 Labeled transition systems

We begin by recalling some basic notions about labeled transition systems.

Definition 5.1 *A* labeled transition system *with divergence* $\langle P, Act, \longrightarrow, \Uparrow\rangle$ *is defined by a set* P *of processes, a set* Act *of atomic actions, a transition relation* $\longrightarrow \subseteq P \times Act \times P$*, and a predicate* $\Uparrow$ *on* P*. The predicate* $\Uparrow$ *is called the* divergence predicate*. The* convergence predicate $\Downarrow$ *on* P *is defined to be the complement of the divergence predicate, that is* $\Downarrow = P \setminus \Uparrow$*. We use* $p \Uparrow$ *and* $p \Downarrow$ *to denote that the process* p *diverges and converges, respectively.*

Transition systems can be used for modeling computations of programming languages [Plo81b] and to identify processes with the same observable behavior. One of the most well-known behavioral equivalences on processes is bisimulation [Mil80, Par81].

Definition 5.2 *Given a transition system* $\langle P, Act, \longrightarrow, \Uparrow\rangle$*, a relation* $R \subseteq P \times P$ *is called a* partial bisimulation *whenever, if* $\langle p, q\rangle \in R$ *then for all* $a \in Act$

1. $p \xrightarrow{a} p' \Rightarrow \exists q' \in P \colon q \xrightarrow{a} q'$ *and* $\langle p', q'\rangle \in R$*;*
2. $p \Downarrow \Rightarrow q \Downarrow$ *and* $(q \xrightarrow{a} q' \Rightarrow \exists p' \in P \colon p \xrightarrow{a} p'$ *and* $\langle p', q'\rangle \in R)$*.*

We write $p \lesssim^B q$ *if there exists a partial bisimulation* R *with* $\langle p, q\rangle \in R$*.*

Partial bisimulations can also be described in terms of iteration [Par81], but in general one needs to consider a non-countable sequence of relations (in the complete lattice $\mathcal{P}(P \times P)$ ordered by subset inclusion) approximating $\lesssim^B$. By considering only countable approximants of $\lesssim^B$ one obtains the so-called *observable equivalence* $\lesssim^\omega = \bigcap_\omega \lesssim^n$ [Mil80], where

- $\lesssim^0 = P \times P$, and
- $p \lesssim^{n+1} q$ if and only if for all $a \in Act$
  1. $p \xrightarrow{a} p' \Rightarrow \exists q' \in P \colon q \xrightarrow{a} q'$ and $p' \lesssim^n q'$;
  2. $p \Downarrow \Rightarrow q \Downarrow$ and $(q \xrightarrow{a} q' \Rightarrow \exists p' \in P \colon p \xrightarrow{a} p'$ and $p' \lesssim^n q')$.

In general for a transition system T, $\lesssim^B \subseteq \lesssim^\omega$. However, if T is image-finite then the two notions coincide [HM85].

A particular example of a transition system is given by the collection of all (finite) *synchronization trees* over an alphabet Act of actions. Define the set $(t \in) ST(Act)$ of finitary synchronization trees over Act by

$$t \quad ::= \quad \Sigma_I\, a_i t_i \mid \Sigma_I\, a_i t_i + \Omega,$$

where I is a finite index set, and all the a_i's are actions in Act for $i \in I$. The set of all finitary synchronization trees can be turned into a transition system $ST(Act) = \langle ST(Act), Act, \longrightarrow, \Uparrow\rangle$, where

- $t \Uparrow$ if and only if Ω is included as a summand of t, and
- $t \xrightarrow{a_i} t_i$ for each summand $a_i t_i$ of t.

Synchronization trees can be used to define a finitary preorder on processes of more general transition systems [Gue81].

Definition 5.3 *For a transition system* $\langle P, Act, \longrightarrow, \Uparrow\rangle$ *define the* finitary preorder $\lesssim^F \subseteq P \times P$ *by*

$$p \lesssim^F q \text{ if and only if } \forall t \in ST(Act)\text{: } t \lesssim^B p \Rightarrow t \lesssim^B q.$$

Since finite synchronization trees are a model for finite processes, the finitary preorder can be considered as the finite observable part of partial bisimulation. For every transition system T, it holds that

$$\lesssim^B \subseteq \lesssim^\omega \subseteq \lesssim^F .$$

In general, these inclusions are strict [Abr91b, page 191].

Another example of a transition system is given by the SFP-domain $\mathcal{D}$ obtained as the initial (and final) solution in the category **SFP** of the recursive domain equation

$$X \quad \cong \quad (\mathbf{1})_\perp \oplus \mathcal{P}_c^{co}\left(\sum_{a \in Act} X\right),$$

where $\mathbf{1}$ is the one-point cpo, Act is a countable set of actions, $(-)_\perp$ is the lift, $\oplus$ is the coalesced sum, $\sum_{a\in Act}$ is the countable separated sum, and $\mathcal{P}_c^{co}(-)$ is the Plotkin powerdomain. Below we will omit the isomorphism pair relating the left and the right hand side of the solution $\mathcal{D}$ of the above domain equation. The SFP-domain $\mathcal{D}$ can be seen as the transition system $\langle \mathcal{D}, Act, \longrightarrow, \Uparrow\rangle$ where

- $d \xrightarrow{a} d'$ if and only if $\langle a, d'\rangle \in d$ and
- $d \Uparrow$ if and only if $\perp \in d$.

The SFP-domain $\mathcal{D}$, seen as a transition system, plays the role of canonical model for the Abramsky's logic for transition systems. Furthermore it can be used as a semantic domain for every transition system modulo the equivalence generated by the finitary preorder [Abr91b].

The order on the domain $\mathcal{D}$ coincides with the bisimulation preorder when $\mathcal{D}$ is seen as transition system. This fact was first proved by Abramsky [Abr91b, Proposition 3.11] using an elementwise characterization of $\mathcal{D}$ as the 'internal colimit' of a sequence of projections. Below, after Lemma 7.2, we will obtain the same result using only the logical interpretation of $\mathcal{D}$.

5.2 Abramsky logic for transition systems

Like the Hennessy-Milner logic [HM85], the idea of Abramsky's infinitary logic $\mathcal{L}_{\infty,\infty}$ for transition systems [Abr91b] is to obtain a suitable characterization of partial bisimulation in terms of a notion of property of processes: $p \lesssim^B q$ if and only if every property satisfied by p is also satisfied by q. However, the finitary restriction of Abramsky's logic differs from the finitary Hennessy-Milner logic in the sense that it characterizes the finitary observable part of partial bisimulation for all transition systems.

Definition 5.4 *Let* $(a \in)Act$ *be a set of actions. The language* $\mathcal{L}_{\infty,\infty}$ *over Act has two sorts:* π *(processes) and* k *(capabilities). We write* $(\phi \in)\mathcal{L}^\pi_{\infty,\infty}$ *for the class of formulae of sort* π*, and* $(\psi \in)\mathcal{L}^k_{\infty,\infty}$ *for the class of formulae of sort* k*, which are defined inductively as follows:*

$$\begin{aligned} \phi \quad &::= \quad \bigvee_I \phi_i \mid \bigwedge_I \phi_i \mid \Box\psi \mid \Diamond\psi \\ \psi \quad &::= \quad \bigvee_I \psi_i \mid \bigwedge_I \psi_i \mid a(\phi), \end{aligned}$$

where I *is an arbitrary index set. If* $I = \emptyset$ *then we write* tt *for* $\bigwedge_I \phi_i$ *and* $\bigwedge_I \psi_i$*, and we write* ff *for* $\bigvee_I \phi_i$ *and* $\bigvee_I \psi_i$*.*

In order to prove properties by induction on the structure of formulae of $\mathcal{L}_{\infty,\infty}$, we define the *height* of a formula as the following ordinal:

$$\begin{array}{lllll}
ht(\bigvee_I \phi_i) & = & ht(\bigwedge_I \phi_i) & = & \sup\{ht(\phi_i) \mid i \in I\} + 1 \\
ht(\Box\psi) & = & ht(\Diamond\psi) & = & ht(\psi) + 1 \\
\\
ht(\bigvee_I \psi_i) & = & ht(\bigwedge_I \psi_i) & = & \sup\{ht(\psi_i) \mid i \in I\} + 1 \\
ht(a(\phi)) & = & ht(\phi). & &
\end{array}$$

For example, $ht(tt) = ht(f\!f) = 1$ and $ht(\Box a(tt) \vee a(\Diamond b(f\!f))) = 2$.

Before we interpret the language $\mathcal{L}_{\infty,\infty}$ we need the following definitions. For a transition system $\langle P, Act, \longrightarrow, \Uparrow\rangle$ define the set Cap of *capabilities* by

$$Cap = \{\bot\} \cup (Act \times P).$$

The set of capabilities of a process $p \in P$ is given by

$$C(p) \quad = \quad \{\bot \mid p \Uparrow\} \cup \{\langle a, q\rangle \mid p \xrightarrow{a} q\}.$$

For a transition system $T = \langle P, Act, \longrightarrow, \Uparrow\rangle$, we interpret the language $\mathcal{L}_{\infty,\infty}$ by means of *satisfaction relations* $\models_\pi \subseteq P \times \mathcal{L}^\pi_{\infty,\infty}$ and $\models_k \subseteq Cap \times \mathcal{L}^k_{\infty,\infty}$ defined as follows:

$$\begin{array}{lcl}
p \models_\pi \bigvee_I \phi_i & \iff & \exists i \in I\colon p \models_\pi \phi_i \\
p \models_\pi \bigwedge_I \phi_i & \iff & \forall i \in I\colon p \models_\pi \phi_i \\
p \models_\pi \Box\phi & \iff & p \Downarrow \text{ and } \forall c \in C(p)\colon c \models_k \phi \\
p \models_\pi \Diamond\phi & \iff & \exists c \in C(p)\colon c \models_k \phi \\
\\
c \models_k \bigvee_I \phi_i & \iff & \exists i \in I\colon c \models_k \phi_i \\
c \models_k \bigwedge_I \phi_i & \iff & \forall i \in I\colon c \models_k \phi_i \\
c \models_k a(\phi) & \iff & c = \langle a, q\rangle \text{ and } q \models_\pi \phi.
\end{array}$$

For a transition system $T = \langle P, Act, \longrightarrow, \Uparrow\rangle$ and formula ϕ of $\mathcal{L}^\pi_{\infty,\infty}$ we write $[\![\phi]\!]^\pi_T$ for $\{p \in P \mid p \models_\pi \phi\}$. *Assertions* A over the language $\mathcal{L}^\sigma_{\infty,\infty}$ are of the form $\phi \leq_\sigma \psi$ or $\phi =_\sigma \psi$ for σ in $\{\pi, k\}$ with ϕ and ψ in $\mathcal{L}^\sigma_{\infty,\infty}$. The satisfaction relation between transition systems T and assertions is defined by

$$\begin{array}{lcl}
T \models \phi \leq_\pi \psi & \iff & \forall p \in P\colon p \models_\pi \phi \text{ implies } p \models_\pi \psi \\
T \models \phi =_\pi \psi & \iff & \forall p \in P\colon p \models_\pi \phi \text{ if and only if } p \models_\pi \psi \\
\\
T \models \phi \leq_k \psi & \iff & \forall c \in Cap\colon c \models_k \phi \text{ implies } c \models_k \psi \\
T \models \phi =_k \psi & \iff & \forall c \in Cap\colon c \models_k \phi \text{ if and only if } c \models_k \psi.
\end{array}$$

As usual, the satisfaction relation can be extended to classes of transition systems $\mathcal{T}$ by

$$\mathcal{T} \models A \quad \iff \quad \forall\, T \in \mathcal{T}\colon T \models A.$$

If $\mathcal{T}$ is the class of all transition systems then we simply write $\models A$.

Let $\mathcal{L}_{\omega,\omega}$ be the sub-language of $\mathcal{L}_{\infty,\infty}$ obtained by the restriction to finite conjunctions and finite disjunctions.

Theorem 5.5 *For a transition system $\langle P, Act, \longrightarrow, \Uparrow\rangle$ and p, q in P,*

(i) *$p \precsim^B q$ if and only if $\forall\phi \in \mathcal{L}^\pi_{\infty,\infty}\colon p \models \phi \Rightarrow q \models \phi$;*

(ii) *$p \precsim^F q$ if and only if $\forall\phi \in \mathcal{L}^\pi_{\omega,\omega}\colon p \models \phi \Rightarrow q \models \phi$.*

Proof: See Theorems 5.6 and 5.8 in [Abr91b]. □

Next we present a proof system for assertions over $\mathcal{L}_{\infty,\infty}$. (We omit the sort subscripts.) The following *logical axioms* give to the language the structure of a large completely distributive lattice.

$$(\leq -ref) \quad \phi \leq \phi$$

$$(\leq -trans) \quad \frac{\phi \leq \psi \text{ and } \psi \leq \chi}{\phi \leq \chi}$$

$$(= -I) \quad \frac{\phi \leq \psi \text{ and } \psi \leq \phi}{\phi = \psi} \qquad\qquad (= -E) \quad \frac{\phi = \psi}{\phi \leq \psi \text{ and } \psi \leq \phi}$$

$$(\wedge - I) \quad \frac{\{\phi \leq \psi_i\}_{i \in I}}{\phi \leq \bigwedge_I \psi_i} \qquad\qquad (\wedge - E) \quad \bigwedge_I \phi_i \leq \phi_k \quad (k \in I)$$

$$(\vee - I) \quad \frac{\{\phi_i \leq \psi\}_{i \in I}}{\bigvee_I \phi_i \leq \psi} \qquad\qquad (\vee - E) \quad \phi_k \leq \bigvee_I \phi_i \quad (k \in I)$$

$$(\wedge - dist) \quad \bigwedge_I \bigvee_{J_i} \phi_{i,j} \quad = \quad \bigvee_{f \in \Phi(\{J_i | i \in I\})} \bigwedge_I \phi_{i,f(i)}.$$

The following *modal axioms* relate constructors with the logical structure.

$$(a- \leq) \quad \frac{\phi \leq \psi}{a(\phi) \leq a(\psi)}$$

$$(a - \wedge) \quad \begin{array}{lll} \text{(i)} & a(\bigwedge_I \phi_i) = \bigwedge_I a(\phi_i) & (I \neq \emptyset) \\ \text{(ii)} & a(\phi) \wedge b(\psi) = \mathit{ff} & (a \neq b) \end{array}$$

$$(a - \vee) \quad a(\bigvee_I \phi_i) = \bigvee_I a(\phi_i)$$

$$(\Box - \leq) \quad \frac{\phi \leq \psi}{\Box\phi \leq \Box\psi}$$

$$(\Box - \wedge) \quad \Box \bigwedge_I \phi_i = \bigwedge_I \Box\phi_i \quad (I \neq \emptyset)$$

$$(\Box - \vee) \quad \Box(\phi \vee \psi) \leq \Box\phi \vee \Diamond\psi$$

$$(\Diamond - \leq) \quad \frac{\phi \leq \psi}{\Diamond\phi \leq \Diamond\psi}$$

$$(\Diamond - \wedge) \quad \Box\phi \wedge \Diamond\psi \leq \Diamond(\phi \wedge \psi)$$

$$(\Diamond - \vee) \quad \Diamond \bigvee_I \phi_i = \bigvee_I \Diamond\phi_i \,.$$

We write $\mathcal{L}_{\infty,\infty} \vdash A$ if the assertion A of $\mathcal{L}_{\infty,\infty}$ is derivable from the above axioms and rules.

Theorem 5.6 *(Soundness) If* $\mathcal{L}_{\infty,\infty} \vdash A$ *then* $\models A$.

Proof: See Theorem 4.2 in [Abr91b]. □

Next we turn to the finitary logic $\mathcal{L}_{\omega,\omega}$ in order to prove the reverse of the above result for the class of all transition systems.

Let $\mathcal{LA}^{\pi}_{\omega,\omega}$ be the *Lindenbaum algebra* of $\mathcal{L}^{\pi}_{\omega,\omega}$, and let $[\phi]$ denote the set of all formulae provably equivalent in $\mathcal{L}^{\pi}_{\omega,\omega}$ to $\phi \in \mathcal{L}^{\pi}_{\omega,\omega}$. The following fundamental result shows that the finitary logic $\mathcal{L}^{\pi}_{\omega,\omega}$ does indeed correspond exactly to the SFP-domain $\mathcal{D}$ taken with the Scott topology.

Theorem 5.7 *Let $\mathcal{KO}(\mathcal{D})$ be the distributive lattice of Scott compact open sets of $\mathcal{D}$ ordered by subset inclusion. The function $\gamma: \mathcal{LA}^{\pi}_{\omega,\omega} \to \mathcal{KO}(\mathcal{D})$ defined, for ϕ in $\mathcal{L}^{\pi}_{\omega,\omega}$, by*

$$\gamma([\phi]) = [\![\phi]\!]^{\pi}_{\mathcal{D}}$$

is a well-defined order isomorphism.

Proof: See Theorem 4.3 in [Abr91b]. □

The proof of the spatiality of the distributive lattice $\mathcal{LA}^{\pi}_{\omega,\omega}$ is equivalent to (strong) completeness of the underlying logical system.

Theorem 5.8 *(Completeness) Let $\mathcal{T}$ be any class of transition systems containing $\mathcal{D}$. For ϕ_1 and ϕ_2 in $\mathcal{L}^{\pi}_{\omega,\omega}$, $\mathcal{T} \models \phi_1 \leq \phi_2$ if and only if $\mathcal{L}^{\pi}_{\omega,\omega} \vdash \phi_1 \leq \phi_2$.*

Proof: For ϕ_1 and ϕ_2 in $\mathcal{L}^{\pi}_{\omega,\omega}$ we have,

$$\begin{array}{lcll}
\mathcal{D} \models_{\pi} \phi_1 \leq_{\pi} \phi_2 & \iff & [\![\phi_1]\!]^{\pi}_{\mathcal{D}} \subseteq [\![\phi_2]\!]^{\pi}_{\mathcal{D}} & \\
& \iff & \gamma([\phi_1]) \subseteq \gamma([\phi_2]) & [\text{definition of } \gamma] \\
& \iff & [\phi_1] \leq [\phi_2] & [\gamma \text{ is an order isomorphism}] \\
& \iff & \mathcal{L}^{\pi}_{\omega,\omega} \vdash \phi_1 \leq_{\pi} \phi_2 & [\text{definition of } \mathcal{LA}^{\pi}_{\omega,\omega}].
\end{array}$$

□

We conclude this section by showing that the SFP-domain $\mathcal{D}$ can be used as semantic domain for all transition systems. Let $T = \langle P, Act, \longrightarrow, \Uparrow \rangle$ be a transition system and let $p \in P$. The set

$$TS(p) = \{[\phi] \in \mathcal{LA}^{\pi}_{\omega,\omega} \mid p \models_{\pi} \phi\}$$

is a prime filter of the distributive lattice $\mathcal{LA}^{\pi}_{\omega,\omega}$. Hence, by Theorem 5.7, it corresponds uniquely to an element in $\mathcal{D}$. Therefore, the assignment $p \mapsto TS(p)$ defines a function $TS[\![\cdot]\!]: P \to \mathcal{D}$ which is unique among all functions $f: P \to \mathcal{D}$ such that

$$p \models_{\pi} \phi \quad \text{if and only if} \quad f(p) \models_{\pi} \phi,$$

for all $p \in P$ and $\phi \in \mathcal{L}^{\pi}_{\omega,\omega}$ [Abr91a, Theorem 5.21]. By the characterization Theorem 5.5, it follows that p and $TS[\![p]\!]$ are equivalent in the finitary preorder $\lesssim^{F}$. Hence the function $TS[\![\cdot]\!]: P \to \mathcal{D}$ can be regarded as a syntax-free semantics which is universal because it is defined for every transition system.

6 Compactly branching transition systems

Theorem 5.8 gives a completeness result for $\mathcal{L}_{\omega,\omega}$. In this section we derive a completeness result for $\mathcal{L}_{\omega,\infty}$, the sub-language of $\mathcal{L}_{\infty,\infty}$ which allows infinite disjunctions but has only finite conjunctions. It is possible to express useful properties in this language that cannot be expressed in $\mathcal{L}_{\omega,\omega}$. Consider properties of a transition system $\langle P, Act, \longrightarrow, \Uparrow \rangle$ like 'the process p converges', 'every a-path starting from p is finite', or 'along every a-path starting from p eventually ψ holds'. The finitary language $\mathcal{L}^{\pi}_{\omega,\omega}$ is too weak to formalize these properties, which however can be expressed in the infinitary language $\mathcal{L}^{\pi}_{\omega,\infty}$ by

- $p \models_{\pi} \Box \bigvee_{a \in Act} a(tt)$;
- $p \models_{\pi} \bigvee_{n \in \omega} \phi_n$, where $\left\{ \begin{array}{lcl} \phi_0 & = & \mathit{ff} \text{ and} \\ \phi_{n+1} & = & \Box(a(\phi_n) \vee \bigvee_{Act \setminus \{a\}} b(tt)); \end{array} \right.$
- $p \models_{\pi} \bigvee_{n \in \omega} \phi_n$, where $\left\{ \begin{array}{lcl} \phi_0 & = & \mathit{ff} \text{ and} \\ \phi_{n+1} & = & \psi \vee (\Diamond a(tt) \wedge \Box(a(\phi_n) \vee \bigvee_{Act \setminus \{a\}} b(tt))). \end{array} \right.$

Adding expressive power to the finitary logic should not change our main motivation for its introduction: it should characterize the finitary observable part of partial bisimulation. We introduce the following scheme over $\mathcal{L}_{\omega,\infty}$ which restricts the class of transition systems and allows to write any formula in $\mathcal{L}_{\omega,\infty}$ as disjunctions (possibly infinite) of finitary formulae in $\mathcal{L}_{\omega,\omega}$:

$$(BN) \quad \Box \bigvee_I \phi_i \leq \bigvee_{J \in Fin(I)} \Box \bigvee_J \phi_j \quad \text{with } \phi_i \in \mathcal{L}_{\omega,\omega} \text{ for each } i \in I\,,$$

where $Fin(I)$ is the set of all finite subsets of I. The intuition behind the above axiom scheme is that of *bounded nondeterminism.* We will see in Lemma 6.4 below that (BN) is equivalent to requiring that the $\Box$ operator distributes over directed joins, a condition that, semantically, corresponds to a statement of compactness (and hence of bounded non-determinism [Plo81a, Smy83]).

A transition system is called *compactly branching* if it satisfies all instances of (BN). It is immediate to see that every weakly finitely branching transition system is compactly branching, where a transition system $T = \langle P, Act, \longrightarrow, \Uparrow\rangle$ is said to be weakly finitely branching if for all $p \in P$ such that $p \Downarrow$ the set

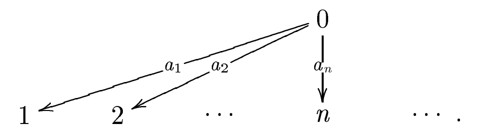
$$Br(p) \quad = \quad \{q \in P \mid \exists a \in Act\colon p \xrightarrow{a} q\}$$

is finite. Since the set of finite synchronization trees is weakly finite branching, it satisfies all instances of (BN).

Clearly not every transition system is compactly branching. For example, for a given enumeration on Act, consider the transition system

$$\langle \mathbb{N}, Act, \longrightarrow, \emptyset\rangle$$

where $0 \xrightarrow{a_n} n$ for $n > 0$, and a_n is the n-th element in the enumeration of Act. Pictorially the above transition system can be represented as follows:



Then

$$0 \models_\pi \Box \bigvee_{n\in\omega} \phi_n \quad \text{where} \quad \begin{cases} \phi_1 & = \quad a_1(tt) \text{ and} \\ \phi_{n+1} & = \quad \phi_n \vee a_n(tt)\,. \end{cases}$$

However, for every $n \geq 1$, $0 \not\models \Box\phi_n$. Hence not every instance of (BN) is a valid axiom for the above transition system.

Next we show that the transition system induced by the SFP-domain $\mathcal{D}$ is compactly branching. Notice that $\mathcal{D}$ is not weakly finite branching.

Lemma 6.1 *The transition system induced by the SFP-domain $\mathcal{D}$ is compactly branching.*

Proof: By the isomorphism of Theorem 5.7 each formula in $\mathcal{L}_{\omega,\omega}$ is interpreted as a Scott compact open subset of $\mathcal{D}$. Hence the validity of all instances of (BN) for $\mathcal{D}$ follows if, for each $d \in \mathcal{D}$, the following holds:

$$d \subseteq \bigcup_I o_i \quad \Rightarrow \quad \exists J \subseteq Fin(I)\colon d \subseteq \bigcup_J o_j\,,$$

where all the o_i's are Scott compact subsets of $\sum_{a\in Act}\mathcal{D}$. The case for $d = \mathbf{1}$ or $\perp$ is immediate. Otherwise d is an element of the Plotkin powerdomain, and hence, by its definition, a Scott compact subset of $\sum_{a\in Act}\mathcal{D}$. Therefore the above statement holds, and $\mathcal{D}$ satisfies all instances of the axiom scheme (BN). $\Box$

Following the steps described in Section 4.1, our next step is to prove that each formula in the extended language is equivalent to a disjunction of formulae of the finitary language.

Lemma 6.2 *(Disjunctive normal form) For every formula ϕ in $\mathcal{L}^{\pi}_{\omega,\infty}$ there exist formulae $\phi_i \in \mathcal{L}^{\pi}_{\omega,\omega}$ with $i \in I$ such that $\mathcal{L}^{\pi}_{\omega,\infty} + (BN) \vdash \phi = \bigvee_I \phi_i$.*

Proof: By induction on the height ht of formulae in $\mathcal{L}_{\omega,\infty}$. □

As an immediate consequence we have the following corollary.

Corollary 6.3 *For every formula ϕ in $\mathcal{L}^{\pi}_{\omega,\infty}$, $\mathcal{L}^{\pi}_{\omega,\infty} + (BN) \vdash \phi = \bigvee\{\psi \in \mathcal{L}^{\pi}_{\omega,\omega} \mid \psi \leq \phi\}$.*

Proof: By Rule $(\vee - I)$ we have $\mathcal{L}^{\pi}_{\omega,\infty} \vdash \bigvee\{\psi \in \mathcal{L}^{\pi}_{\omega,\omega} \mid \psi \leq \phi\} \leq \phi$. The other direction follows because by Lemma 6.2 there exists $\phi_i \in \mathcal{L}^{\pi}_{\omega,\omega}$ with $i \in I$ such that $\mathcal{L}^{\pi}_{\omega,\infty} + (BN) \vdash \phi = \bigvee_I \phi_i$ and hence $\mathcal{L}^{\pi}_{\omega,\infty} + (BN) \vdash \phi_i \leq \phi$ for all $i \in I$. □

The above lemma together with the soundness Theorem 5.6, the definition of the satisfaction relation, and the characterization Theorem 5.5, imply that for compactly branching transition systems $\langle P, Act, \longrightarrow, \Uparrow\rangle$ and processes p, q in P,

$$p \lesssim^F q \quad \text{if and only if} \quad \forall \phi \in \mathcal{L}^{\pi}_{\omega,\infty}\colon p \models \phi \Rightarrow q \models \phi.$$

We are now ready to give a topological characterization of compactly branching transition systems. Let $T = \langle P, Act, \longrightarrow, \Uparrow\rangle$ be a transition system and let $\mathcal{O}(T)$ denote the set of all $[\![\phi]\!]^{\pi}_T$ for ϕ in $\mathcal{L}^{\pi}_{\omega,\infty}$. Clearly, $\mathcal{O}(T)$ forms a topology on P. Transition systems together with a topology are introduced in the context of modal logic in [Esa74], where, in a restricted form, the implication from (iii) to (ii) of the next lemma is proved (the proof of the other direction can been found in [BK95]).

Lemma 6.4 *For a transition system $T = \langle P, Act, \longrightarrow, \Uparrow\rangle$ the following are equivalents:*

(i) *it satisfies all instances of the axiom scheme (BN);*

(ii) *for all $p \in P$ such that $p \Downarrow$, the set*

$$Br(p) \quad = \quad \{q \in P \mid \exists a \in Act\colon p \xrightarrow{a} q\}$$

is compact in the topology $\mathcal{O}(T)$;

(iii) *it satisfies all instances of the following axiom scheme:*

$$(BN') \quad \Box \bigvee_I \phi_i \leq \bigvee_{J \in Fin(I)} \Box \bigvee_J \phi_j \quad \text{with } \phi_i \in \mathcal{L}_{\omega,\infty} \text{ for each } i \in I\,.$$

Proof: Clearly every instance of (BN) is an instance of (BN'). Hence (iii) implies (i). In order to prove (i) implies (ii) assume T satisfies (BN). Take a $p \in P$ with $p \Downarrow$ and $Br(p) \subseteq \bigcup_I [\![\phi_i]\!]^{\pi}_T$, where $\phi_i \in \mathcal{L}_{\omega,\omega}$ for each $i \in I$. Then $p \models_{\pi} \Box \bigvee_I \phi_i$. Hence, by (BN), $p \models_{\pi} \bigvee_{J \in Fin(I)} \Box \bigvee_J \phi_j$, that is, $Br(p) \subseteq \bigcup_J [\![\phi_j]\!]^{\pi}_T$ for a finite subset J of I. By Lemma 6.2, the soundness Theorem 5.6, and because T satisfies (BN), the interpretations of formulae in $\mathcal{L}_{\omega,\omega}$ form a basis for $\mathcal{O}(T)$. Hence every cover of $Br(p)$ by basic opens has a finite subcover, from which it follows that $Br(p)$ is compact in $\mathcal{O}(T)$.

It remains to prove that (ii) implies (iii). Assume that if $p \Downarrow$ then the set $Br(p)$ is compact in the topology $\mathcal{O}(T)$, and let $p \models_{\pi} \Box \bigvee_I \phi_i$, where ϕ_i in $\mathcal{L}_{\omega,\infty}$ for each $i \in I$. Since p converges, $Br(p) \subseteq \bigcup_I [\![\phi_i]\!]^{\pi}_T$. But $Br(p)$ is compact, hence $Br(p) \subseteq \bigcup_J [\![\phi_j]\!]^{\pi}_T$ for some finite subset J of I. It follows that p satisfies (BN'). □

The next step is to prove the completeness of the logic $\mathcal{L}^{\pi}_{\omega,\infty}$ for the class of compactly branching transition systems. We proceed as for the finite case: let $\mathcal{LA}^{\pi}_{\omega,\infty}$ be the Lindenbaum algebra of $\mathcal{L}^{\pi}_{\omega,\infty}$ with as elements equivalence classes of formulae provably equivalent in $\mathcal{L}^{\pi}_{\omega,\infty} + (BN)$. The poset $\mathcal{LA}^{\pi}_{\omega,\infty}$ is a frame with meets and joins defined as expected.

Lemma 6.5 *The frame $\mathcal{LA}^{\pi}_{\omega,\infty}$ is free over the distributive lattice $\mathcal{LA}^{\pi}_{\omega,\omega}$.*

Proof: For any frame F and function $f\colon \mathcal{LA}^{\pi}_{\omega,\omega} \to F$ preserving finite meets and finite joins, define $h\colon \mathcal{LA}^{\pi}_{\omega,\infty} \to F$ by

$$h([\phi]) \quad = \quad \bigvee \{f([\psi]) \mid \psi \in \mathcal{L}^{\pi}_{\omega,\omega} \text{ and } \psi \leq \phi\}\,.$$

By definition, $h([\psi]) = f([\psi])$ for all ψ in $\mathcal{L}^{\pi}_{\omega,\omega}$.

In order to prove that h preserves arbitrary joins first we note that for $\psi \in \mathcal{L}^{\pi}_{\omega,\omega}$ and $\phi_i \in \mathcal{L}^{\pi}_{\omega,\infty}$ with $i \in I$,

$$\psi \leq \bigvee_I \phi_i \quad \text{if and only if} \quad \exists J \in \mathit{Fin}(I)\,.\psi \leq \bigvee_J \psi_j\,. \tag{5}$$

The implication from right to left is immediate. To prove the other direction we can use Corollary 6.3 in order to restrict our attention only to formulae $\phi_i \in \mathcal{L}^{\pi}_{\omega,\omega}$. Because the SFP-domain $\mathcal{D}$ is compactly branching we have

$$\psi \leq \bigvee_I \phi_i \quad \Rightarrow \quad [\![\psi]\!]^{\pi}_{\mathcal{D}} \subseteq [\![\bigvee_I \phi_i]\!]^{\pi}_{\mathcal{D}} = \bigcup_I [\![\phi_i]\!]^{\pi}_{\mathcal{D}}\,.$$

By Theorem 5.7, $[\![\psi]\!]^{\pi}_{\mathcal{D}}$ and $[\![\phi_i]\!]^{\pi}_{\mathcal{D}}$, for all $i \in I$, are compact open subsets of $\mathcal{D}$. Hence there exists a finite subset J of I such that

$$[\![\psi]\!]^{\pi}_{\mathcal{D}} \subseteq \bigcup_J [\![\phi_j]\!]^{\pi}_{\mathcal{D}} = [\![\bigvee_J \phi_j]\!]^{\pi}_{\mathcal{D}}\,.$$

By the completeness Theorem 5.8 it follows that $\psi \leq \bigvee_J \phi_j$.

Now we can prove that h preserves directed joins. Let $S \subseteq \mathcal{LA}^{\pi}_{\omega,\infty}$ be directed. We have

$$\begin{aligned}
h([\bigvee S]) \quad &= \quad \bigvee \{f([\psi]) \mid \psi \in \mathcal{L}^{\pi}_{\omega,\omega} \text{ and } \psi \leq \bigvee S\} \quad \text{[definition of } h\text{]} \\
&= \quad \bigvee \{f([\psi]) \mid \psi \in \mathcal{L}^{\pi}_{\omega,\omega} \text{ and } \exists \phi \in S\,.\psi \leq \phi\} \quad \text{[property (5)]} \\
&= \quad \bigvee \{h([\phi]) \mid \phi \in S\} \quad \text{[definition of } h\text{]}.
\end{aligned}$$

Preservation of finite joins is immediate. Hence h preserves arbitrary joins.

Next we prove that h preserves finite meets. We use ψ, ψ' and ψ'' to denote formulae ranging over $\mathcal{L}^{\pi}_{\omega,\omega}$. For formulae ϕ' and ϕ'' in $\mathcal{L}^{\pi}_{\omega,\infty}$ we have

$$\begin{aligned}
h([\phi'] \wedge [\phi'']) \quad &= \quad \bigvee \{f([\psi]) \mid [\psi] \leq [\phi'] \wedge [\phi'']\} \quad \text{[definition of } h\text{]} \\
&= \quad \bigvee \{f([\psi'] \wedge [\psi'']) \mid [\psi'] \leq [\phi'] \text{ and } [\psi''] \leq [\phi'']\} \quad \text{[easy calculation]} \\
&= \quad \bigvee \{f([\psi']) \wedge f([\psi'']) \mid [\psi'] \leq [\phi'] \text{ and } [\psi''] \leq [\phi'']\} \quad [f \text{ preserves meets]} \\
&= \quad \bigvee \{f([\psi']) \mid [\psi'] \leq [\phi']\} \wedge \bigvee \{f([\psi'']) \mid [\psi''] \leq [\phi'']\} \quad \text{[distributivity]} \\
&= \quad h([\phi']) \wedge h([\phi'']) \quad \text{[definition of } h\text{]}.
\end{aligned}$$

By Corollary 6.3 and the definition of h it follows that h is the unique frame morphism such that $h \circ \iota = f$, where $\iota\colon \mathcal{LA}^{\pi}_{\omega,\omega} \to \mathcal{LA}^{\pi}_{\omega,\infty}$ is the obvious inclusion function. $\Box$

We can now lift the isomorphism of Lemma 5.7 to an isomorphism which maps formulae of $\mathcal{L}^{\pi}_{\omega,\infty}$ to Scott open sets of $\mathcal{D}$.

Lemma 6.6 *Let $O(\mathcal{D})$ be the frame of Scott open subsets of $\mathcal{D}$. The assignment $[\phi] \mapsto [\![\phi]\!]^{\pi}_{\mathcal{D}}$ defines a unique order isomorphism $\gamma^{+}\colon \mathcal{LA}^{\pi}_{\omega,\infty} \to \mathcal{O}(\mathcal{D})$ such that $\gamma^{+}([\phi]) = \gamma([\phi])$ for all ϕ in $\mathcal{L}^{\pi}_{\omega,\omega}$.*

Proof: Because $\mathcal{D}$ is an SFP-domain, when taken with its Scott topology it forms a spectral space. Hence, by Theorem 3.8, the lattice of Scott open sets $\mathcal{O}(\mathcal{D})$ is the free frame over the distributive lattice of Scott compact open sets $\mathcal{KO}(\mathcal{D})$. Furthermore, the latter is, by Lemma 5.7, order isomorphic to the Lindenbaum algebra $\mathcal{LA}^{\pi}_{\omega,\omega}$. But $\mathcal{LA}^{\pi}_{\omega,\infty}$ is the free frame over the distributive lattice $\mathcal{LA}^{\pi}_{\omega,\omega}$ (Lemma 6.5), hence $\mathcal{O}(\mathcal{D})$ is order isomorphic to $\mathcal{LA}^{\pi}_{\omega,\infty}$. The isomorphism is given by the unique extension γ^+ of the function $\gamma\colon \mathcal{LA}^{\pi}_{\omega,\omega} \to \mathcal{KO}(\mathcal{D})$ given in Theorem 5.7. For all ϕ in $\mathcal{L}^{\pi}_{\omega,\infty}$, it can be characterized by

$$\begin{aligned}
\gamma^+([\phi]) &= \gamma^+(\bigvee\{[\psi] \mid \psi \in \mathcal{L}^{\pi}_{\omega,\omega} \text{ and } \psi \leq \phi\}) \quad \text{[Corollary 6.3]}\\
&= \bigcup\{\gamma([\psi]) \mid \psi \in \mathcal{L}^{\pi}_{\omega,\omega} \text{ and } \psi \leq \phi\} \quad [\gamma^+ \text{ preserves arbitrary joins and comutativity}]\\
&= \bigcup\{[\![\psi]\!]^{\pi}_{\mathcal{D}} \mid \psi \in \mathcal{L}^{\pi}_{\omega,\omega} \text{ and } \psi \leq \phi\} \quad \text{[Theorem 5.7]}\\
&= [\![\bigvee\{\psi \mid \psi \in \mathcal{L}^{\pi}_{\omega,\omega} \text{ and } \psi \leq \phi\}]\!]^{\pi}_{\mathcal{D}} \quad [\text{definition of } [\![-]\!]^{\pi}_{\mathcal{D}}]\\
&= [\![\phi]\!]^{\pi}_{\mathcal{D}} \quad [\mathcal{D} \text{ is compactly branching}].
\end{aligned}$$

□

Soundness of the logical system associated to $\mathcal{L}^{\pi}_{\omega,\infty}$ extended with the scheme (BN) follows from Theorem 5.6 and the definition of compactly branching transition systems. In a way similar to the completeness Theorem 5.8, completeness follows from the duality Lemma 6.6.

Theorem 6.7 *(Completeness) Let $\mathcal{CB}$ be any class of compactly branching transition systems containing $\mathcal{D}$. For ϕ_1 and ϕ_2 in $\mathcal{L}^{\pi}_{\omega,\infty}$, $\mathcal{CB} \models \phi_1 \leq \phi_2$ if and only if $\mathcal{L}^{\pi}_{\omega,\infty} + (BN) \vdash \phi_1 \leq \phi_2$.* □

An immediate consequence of Lemma 6.4 and the above completeness result is that each instance of the axiom scheme (BN') is provable in $\mathcal{L}_{\omega,\infty}$ extended with the axiom scheme (BN).

7 Finitary transition systems

The language $\mathcal{L}_{\omega,\infty}$ is more expressive than the finitary language $\mathcal{L}_{\omega,\omega}$. Next we consider the even more expressive language $\mathcal{L}_{\infty,\infty}$. For example, given a transition system $\langle P, Act, \longrightarrow, \Uparrow\rangle$ we can specify in $\mathcal{L}_{\infty,\infty}$ properties like 'there exists an infinite a-path starting from the process p', and 'at any point of any path starting from p an a-transition is always possible', respectively by

- $p \models_\pi \bigwedge_{n\in\omega} \phi_n$, where $\begin{cases} \phi_0 &= tt \text{ and} \\ \phi_{n+1} &= \Diamond a(\phi_n); \end{cases}$
- $p \models_\pi \bigwedge_{n\in\omega} \phi_n$, where $\begin{cases} \phi_0 &= tt \text{ and} \\ \phi_{n+1} &= \Diamond a(\phi_n) \wedge \bigwedge_{Act}(\Box b(\phi_n) \vee \bigvee_{Act\setminus\{b\}} c(tt)). \end{cases}$

By using the new characterization of sober spaces given in Theorem 3.6, we will now prove a completeness result for $\mathcal{L}_{\infty,\infty}$ following the same pattern as for the completeness result of $\mathcal{L}_{\omega,\infty}$.

First we introduce two finitary axiom schemes over $\mathcal{L}_{\infty,\infty}$ in order to prove a normal form for the formulae in the language. These schemes are necessary for the domain $\mathcal{D}$ to be a sound and complete model of $\mathcal{L}_{\infty,\infty}$. However they will also restrict the class of transition systems under consideration. The two axiom schemes are

$$\begin{array}{lll}
(BN) & \Box \bigvee_I \phi_i \leq \bigvee_{J\in Fin(I)} \Box \bigvee_J \phi_j & \text{with } \phi_i \in \mathcal{L}_{\omega,\omega} \text{ for each } i \in I\\
(FA) & \bigwedge_{J\in Fin(I)} \Diamond \bigwedge_J \phi_j \leq \Diamond \bigwedge_I \phi_i & \text{with } \phi_i \in \mathcal{L}_{\omega,\omega} \text{ for each } i \in I\,,
\end{array}$$

where, as before, $Fin(I)$ is the set of all finite subsets of I. The axiom scheme (FA) is the dual of (BN). While the axiom (BN) is related to the width of a computation, the axiom (FA) is related to its length. The latter is analogous to the requirement that we cannot distinguish a set from its closure by means of compact open sets [Fin73] (thinking of each ϕ_i as a compact open set, or, equivalently, as a finite observable property). It can be understood as a notion of *finite approximation.*

For example, the transition system induced by the set of finite synchronization trees satisfies all instances of the two axiom schemes above. In general, a transition system which satisfies all instances of (BN) and (FA) is called *finitary.*

We have already seen in the previous section an example of a transition system that does not satisfy (BN). Consider now the transition system $\langle \mathbb{N}\times\mathbb{N}, Act, \longrightarrow, \{\langle 0,0\rangle\}\rangle$ where $\langle 0,0\rangle \xrightarrow{a} \langle 1,n\rangle$ for all $n \geq 0$ and $\langle n,m\rangle \xrightarrow{a} \langle n+1,m\rangle$ if $n \leq m$. Pictorially the above transition system can be represented as follows:

$$\begin{tikzcd}
& & & \langle 0,0\rangle \arrow[dlll, "a"'] \arrow[dll, "a"'] \arrow[d, "a"] & \\
\langle 1,0\rangle & \langle 1,1\rangle \arrow[d, "a"] & \cdots & \langle 1,n\rangle \arrow[d, "a"] & \cdots . \\
& \langle 2,1\rangle & & \langle 2,n\rangle \arrow[d, "a"] & \\
& & & \vdots \arrow[d, "a"] & \\
& & & \langle n+1,n\rangle &
\end{tikzcd}$$

Then, for every finite subset J of ω,

$$\langle 0,0\rangle \models_\pi \Diamond \bigwedge_J \phi_j \quad \text{where} \quad \left\{ \begin{array}{lcl} \phi_1 & = & tt \text{ and} \\ \phi_{n+1} & = & a(\Diamond\phi_n)\,. \end{array} \right.$$

However $\langle 0,0\rangle \not\models \Diamond \bigwedge_\omega \phi_n$. Hence the above transition system does not satisfy (FA). What is 'missing' is a branch with an infinite sequence of transitions all labeled by a.

Next we recall that the transition system induced by the SFP-domain $\mathcal{D}$ is finitary.

Lemma 7.1 *The transition system induced by the SFP-domain $\mathcal{D}$ is finitary.*

Proof: See Theorem 5.15 in [Abr91b]. $\square$

Semantically, finitary transitions systems are exactly those transition systems for which $\lesssim^F$ and $\lesssim^B$ coincide (see Lemma 7.4). Logically, the axioms (BN) and (FA) allow us to rewrite a formula in $\mathcal{L}^\pi_{\infty,\infty}$ as a conjunction of disjunctions of formulae in $\mathcal{L}_{\omega,\omega}$. This fact will be essential in the proof of our completeness result.

Lemma 7.2 *For each ϕ in $\mathcal{L}^\pi_{\infty,\infty}$ there exist formulae $\phi_i \in \mathcal{L}^\pi_{\omega,\infty}$, $i \in I$, such that $\mathcal{L}^\pi_{\infty,\infty}+(BN)+(FA) \vdash \phi =_\pi \bigwedge_I \phi_i$.*

Proof: By induction on the height ht of formulae in $\mathcal{L}_{\infty,\infty}$. See also Lemma 5.17 of [Abr91b]. $\square$

The above lemma implies that, for every formula $\phi \in \mathcal{L}^\pi_{\infty,\infty}$, $\mathcal{L}^\pi_{\infty,\infty} + (BN) + (FA) \vdash \phi = \bigwedge\{\psi \in \mathcal{L}^\pi_{\omega,\infty} \mid \phi \leq \psi\}$.

Another immediate consequence of the above lemma is the following characterization property. For a finitary transition system $\langle P, Act, \longrightarrow, \Uparrow\rangle$ and p, q in P,

$$p \lesssim^F q \quad \text{if and only if} \quad \forall\phi \in \mathcal{L}^\pi_{\infty,\infty}\colon p \models \phi \Rightarrow q \models \phi.$$

By Theorem 5.5 it follows that for finitary transition systems, $\lesssim^F$ and $\lesssim^B$ coincide, while, by the duality Theorem 5.7, it follows that the order of $\mathcal{D}$, which is equivalent to the specialization order induced by Scott topology $\mathcal{O}(\mathcal{D})$ [GHK$^+$80, Remark II.1.4], coincides with the finitary preorder

$\lesssim^F$. Therefore, in $\mathcal{D}$, $d_1 \leq d_2$ if and only if $d_1 \lesssim^B d_2$, that is, $\mathcal{D}$ is internally fully abstract with respect to partial bisimulation.

Next we show that we can strengthen the conditions for finitary transition systems a bit more. This lemma is the equivalent of Lemma 6.4 for compactly branching transition systems.

Lemma 7.3 *A transition system $T = \langle P, Act, \longrightarrow, \Uparrow\rangle$ satisfies all instances of the axiom schemes (BN) and (FA) if and only if it satisfies the following axiom schemes:*

$$\begin{array}{lll} (BN') & \Box \bigvee_I \phi_i \leq \bigvee_{J \in Fin(I)} \Box \bigvee_J \phi_j & \textit{with } \phi_i \in \mathcal{L}_{\omega,\infty} \textit{ for each } i \in I \\ (FA') & \bigwedge_{J \in Fin(I)} \Diamond \bigwedge_J \phi_j \leq \Diamond \bigwedge_I \phi_i & \textit{with } \phi_i \in \mathcal{L}_{\omega,\infty} \textit{ for each } i \in I\,. \end{array}$$

Proof: If T satisfies all instances of (BN') and (FA') then clearly it satisfies also all instances of (BN) and (FA). Conversely, assume T satisfies all instances of (BN) and (FA). By Lemma 6.4 T satisfies all instances of (BN').

Recall now that $\mathcal{O}(T)$ denotes the topology with open sets of the form $[\![\phi]\!]^\pi_T$ for ϕ in the restricted language $\mathcal{L}^\pi_{\omega,\infty}$. By Lemma 6.2, the soundness Theorem 5.6, and because T satisfies (BN), the interpretations of formulae in $\mathcal{L}_{\omega,\omega}$ form a basis for $\mathcal{O}(T)$.

Assume for some set I that $p \cap \bigcap_J o_j \neq \emptyset$ for all finite subsets J of I, where $o_i \in \mathcal{O}(T)$ for all $i \in I$. We need to prove that $p \cap \bigcap_I o_i \neq \emptyset$. Since J is finite, $\bigcap_J o_j$ is an open set for all $J \in Fin(I)$. Hence, for each $J \in Fin(I)$, there exists a basic open u_J subset of $\bigcap_J o_j$ such that $p \cap u_J \neq \emptyset$. Furthermore, by definition of basic open, $u_J = [\![\phi]\!]^\pi_T$ for some formula ϕ in $\mathcal{L}_{\omega,\omega}$. Because T satisfies (FA) it follows that $p \cap \bigcap_I u_i \neq \emptyset$, where $u_i = u_J$ if $i \in J$. By construction $u_i \subseteq o_i$ for all $i \in I$, hence also $p \cap \bigcap_I o_i \neq \emptyset$. It follows that p satisfies all instances of (FA'). $\Box$

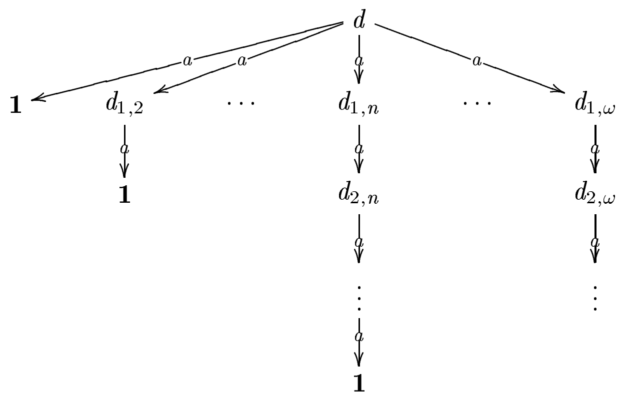
In general a finitary transition system *does not satisfy* the stronger axiom scheme where we allow formulae ϕ_i to be in $\mathcal{L}_{\infty,\infty}$:

$$(BN'') \quad \Box \bigvee_I \phi_i \leq \bigvee_{J \in Fin(I)} \Box \bigvee_J \phi_j \quad \text{with } \phi_i \in \mathcal{L}_{\infty,\infty} \text{ for each } i \in I.$$

Indeed, consider the finitary transition system $\mathcal{D}$, and let $d = \{\langle a, d_{1,m}\rangle \mid m \in \mathbb{N} \cup \{\omega\}\}$, where

$$d_{n,m} \quad = \quad \begin{cases} \mathbf{1} & \text{if } n = m \\ \{\langle a, d_{n+1,m}\rangle\} & \text{if } n < m. \end{cases}$$

The set d is an element of $\mathcal{D}$ [Abr91b, page 199] as it can be defined as the least fixed point of a continuous function from $\mathcal{D}$ to $\mathcal{D}$. Pictorially d can be represented as the following transition system:



Consider the formula

$$\Box \bigvee_{m \in \mathbb{N} \cup \{\omega\}} a(\bigwedge_{n<m} \phi_n)$$

where $\phi_0 = \Box a(f\!f)$ and $\phi_{n+1} = \Box a(\phi_n)$. Note that $a(\bigwedge_{n<\omega} \phi_n)$ is a formula in $\mathcal{L}_{\infty,\infty}$ but not in $\mathcal{L}_{\omega,\infty}$. Informally, the formula ϕ is satisfied by a process p of a transition system only if p

converges and every path starting from p is a non-trivial a-path (possibly infinite). For example, the above d satisfies ϕ because d converges and every path starting from d is an infinite or finite a-path of length greater than or equal to one. If the finitary transition system $\mathcal{D}$ were to satisfy all instances of the axiom scheme (BN'') then d would satisfy also the formula

$$\bigvee_{J \in Fin(\mathbb{N} \cup \{\omega\})} \Box \bigvee_{m \in J} a(\bigwedge_{n<m} \phi_n) .$$

But this is not the case, because for every finite subset J of $\mathbb{N} \cup \{\omega\}$ we can always find an a-path starting from d with a length different from any $m \in J$.

Following the line of proof of Lemma 6.4, it is not hard to see that a transition system $T = \langle P, Act, \longrightarrow, \Uparrow \rangle$ satisfies all instances of the axiom scheme (BN'') if and only if for all convergent $p \in P$, the set $Br(p)$ is compact in the Alexandroff topology of P taken with the preorder $\lesssim^F$.

Finitary transition systems can also be characterized in terms of partial bisimulation as follows.

Lemma 7.4 *For any transition system $T = \langle P, Act, \longrightarrow, \Uparrow \rangle$ the following conditions are equivalent:*

1. *T is finitary,*
2. *for all $p \in P$, p and $TS[\![p]\!]$ are equivalent in the bisimulation preorder $\lesssim^B$,*
3. *the finitary preorder $\lesssim^F$ coincides with bisimulation preorder $\lesssim^B$ in the transition system obtained as the disjoint union of T and $\mathcal{D}$.*

Proof: See Lemma 5.22 of [Abr91b]. □

In the last condition of the above lemma we need to consider the disjoint union of T and $\mathcal{D}$ because T alone may not have enough processes to prove the equivalence between $\lesssim^F$ and $\lesssim^B$.

To prove the completeness of the logic $\mathcal{L}^{\pi}_{\infty,\infty}$ for the class of finitary transition systems, consider its Lindenbaum algebra $\mathcal{LA}^{\pi}_{\infty,\infty}$ with as elements equivalence classes of formulae provably equivalent in $\mathcal{L}^{\pi}_{\infty,\infty} + (BN) + (FA)$. The logical axioms say that the poset $\mathcal{LA}^{\pi}_{\infty,\infty}$ is a completely distributive lattice. By Lemma 7.2 and with a proof similar to the proof of Lemma 6.5, it is not hard to see that $\mathcal{LA}^{\pi}_{\infty,\infty}$ enjoys universal properties.

Lemma 7.5 *The completely distributive lattice $\mathcal{LA}^{\pi}_{\infty,\infty}$ is free over the frame $\mathcal{LA}^{\pi}_{\omega,\infty}$.* □

By Theorem 3.2 it follows that the inclusion function

$$\iota : \mathcal{LA}^{\pi}_{\omega,\infty} \hookrightarrow \mathcal{LA}^{\pi}_{\infty,\infty}$$

is the free observation frame over $\mathcal{LA}^{\pi}_{\omega,\infty}$.

Lemma 7.6 *Let $\mathcal{Q}(\mathcal{D})$ be the completely distributive lattice of saturated subsets of $\mathcal{D}$ with respect to the Scott topology on $\mathcal{D}$. The assignment $[\phi] \mapsto [\![\phi]\!]^{\pi}_{\mathcal{D}}$ defines the unique order isomorphism $\gamma^{\star} : \mathcal{LA}^{\pi}_{\infty,\infty} \to \mathcal{Q}(\mathcal{D})$ such that $\gamma^{\star}([\phi]) = \gamma([\phi])$ for all $\phi \in \mathcal{L}^{\pi}_{\omega,\omega}$.*

Proof: Because $\mathcal{D}$ is an SFP-domain, if it is equipped with the Scott topology then it forms a sober space. Hence, by Theorem 3.6, the lattice of saturated sets $\mathcal{Q}(\mathcal{D})$ is the free completely distributive lattice over the frame of Scott open sets $\mathcal{O}(\mathcal{D})$, which, by Lemma 6.6, is order isomorphic to the Lindenbaum algebra $\mathcal{LA}^{\pi}_{\omega,\infty}$. But $\mathcal{LA}^{\pi}_{\infty,\infty}$ is the free completely distributive lattice over the frame $\mathcal{LA}^{\pi}_{\omega,\infty}$ (Lemma 7.5), and hence $\mathcal{Q}(\mathcal{D})$ is order isomorphic to $\mathcal{LA}^{\pi}_{\infty,\infty}$. The isomorphism is given by the unique extension $\gamma^{\star}$ of the function $\gamma^{+} : \mathcal{LA}^{\pi}_{\omega,\infty} \to \mathcal{O}(\mathcal{D})$ which can be characterized by

$$\begin{aligned}
\gamma^{\star}([\phi]) &= \gamma^{\star}(\bigwedge\{[\psi] \mid \psi \in \mathcal{L}^{\pi}_{\omega,\infty} \text{ and } \phi \leq \psi\}) \quad \text{[Lemma 6.2]} \\
&= \bigcap\{\gamma^{+}([\phi]) \mid \psi \in \mathcal{L}^{\pi}_{\omega,\infty} \text{ and } \phi \leq \psi\} \quad [\gamma^{\star} \text{ preserves meets and comutativity}] \\
&= \bigcap\{[\![\phi]\!]^{\pi}_{\mathcal{D}} \mid \psi \in \mathcal{L}^{\pi}_{\omega,\infty} \text{ and } \phi \leq \psi\} \quad \text{[Theorem 6.6]} \\
&= [\![\bigwedge\{\psi\}]\!]^{\pi}_{\mathcal{D}} \mid \psi \in \mathcal{L}^{\pi}_{\omega,\infty} \text{ and } \phi \leq \psi\}]\!]^{\pi}_{\mathcal{D}} \quad [\text{definition of } [\![-]\!]^{\pi}_{\mathcal{D}}] \\
&= [\![\phi]\!]^{\pi}_{\mathcal{D}} \quad [\mathcal{D} \text{ is finitary}].
\end{aligned}$$

□

As before, soundness of the logical system associated with $\mathcal{L}^{\pi}_{\omega,\infty}$ including both the finitary schemes (BN) and (FA) follows from Theorem 5.6 and from the definition of finitary transition systems. In a similar way to the proof of the completeness Theorem 5.8, completeness follows from the above duality result.

Theorem 7.7 *(Completeness) Let $\mathcal{FT}$ be any class of finitary transition systems containing $\mathcal{D}$. For ϕ_1 and ϕ_2 in $\mathcal{L}^{\pi}_{\infty,\infty}$, $\mathcal{FT} \models \phi_1 \leq \phi_2$ if and only if $\mathcal{L}^{\pi}_{\infty,\infty} + (BN) + (FA) \vdash \phi_1 \leq \phi_2$.* □

As consequence of Lemma 7.3 and the above completeness result, each instance of the axiom schemes (BN') and (FA') is provable in $\mathcal{L}_{\infty,\infty}$ extended with the axiom schemes (BN) and (FA).

8 Conclusion

In this paper we have given a new characterization of sober spaces which can be used for an infinitary extension of every logic based on a topological interpretation, and in particular for an infinitary extension of Abramsky's logic of domains. We have treated an example of infinitary logic for a particular domain involving the Plotkin powerdomain construction. An infinitary logical interpretation of the whole typed language proposed by Abramsky (including the function space construction) will be presented elsewhere. In this paper we concentrated on one example to illustrate the general technique.

Our main motivation for the introduction of an infinitary domain logic as a specification formalism is not to improve over the known specification tools but rather to analyse them by means of general and reusable mathematical notions from topology and domain theory (examples in this direction include a domain logic for Gamma [GH94] which was originally formulated as a transition assertion logic [EHJ93], and a domain logic for a shared-variable parallel language [Zha91] which was originally formulated by Brookes [Bro85]). This is part of Abramsky's general programme of connecting domain theory and operational notions of observability with denotational semantics and program logics.

The present paper does not deal with a formal comparison between Abramsky's logic and Hennessy-Milner logic for transition systems. Such a comparison can be found in [Abr91b], where $\mathcal{L}_{\infty,\infty}$ is proved equivalent to the infinitary Hennessy-Milner logic in the sense that a process of a transition system satisfies a formula of Abramsky's logic if and only if it satisfies the equivalent formula in the Hennessy-Milner logic. Hence formulae of the infinitary Hennessy-Milner logic are interpreted as saturated sets of the SFP-domain $\mathcal{D}$. However it remains an open problem to give axioms and rules for the infinitary Hennessy-Milner logic such that this interpretation is an order pre-isomorphism.

An intriguing exercise that we leave for future work is to see whether the compact ultrametric space introduced by De Bakker and Zucker [BZ82] as unique solution of the domain equation

$$X \quad \cong \quad \mathcal{P}_{co}(Act \times \frac{1}{2} \cdot X).$$

is finitary when interpreted as transition system [GR89].

References

[Abr87] S. Abramsky. *Domain theory and the logic of observable properties.* PhD thesis, Queen Mary College, University of London, 1987.

[Abr91a] S. Abramsky. Domain theory in logical form. *Annals of Pure and Applied Logic*, 51(1):1–77, 1991.

[Abr91b] S. Abramsky. A domain equation for bisimulation. *Information and Computation*, 92(2):161–218, 1991.

[AJ94] S. Abramsky and A. Jung. Domain theory. In S. Abramsky, D.M. Gabbay, and T.S.E. Maibaum, editors, *Handbook of Logic in Computer Science, volume III - Semantic Structures*. Clarendon Press, 1994.

[BK95] M.M. Bonsangue and M.Z. Kwiatkowska. Re-interpreting the modal μ-calculus. In A. Ponse, M. de Rijke, and Y. Venema, editors, *Modal Logic and Process Algebra*, volume 53 of *CSLI Lecture notes*, pages 65–83, Stanford, 1995. Centre for Study of Languages and Information.

[BK97] M.M. Bonsangue and J.N. Kok. Infinitary domain logics for finitary transition systems. In M. Abadi and T. Ito, editors, *Proceedings of TACS'97, Sendai, Japan*, volume 1281 of *Lecture Notes in Computer Science*, pages 213–232. Springer-Verlag, 1997.

[BJK95] M.M. Bonsangue, B. Jacobs, and J.N. Kok. Duality beyond sober spaces: topological spaces and observation frames. *Theoretical Computer Science*, 151(1):79–124, 1995.

[Bon97] M.M. Bonsangue. *Topological Dualities in Semantics*. Volume 8 of *Electronic Notes in Theoretical Computer Science*. Elsevier Science, 1997. Available from URL `http://www.elsevier.nl/locate/entcs`.

[Bro85] S.D. Brookes. An axiomatic treatment of a parallel programming language. In R. Parikh, editor, *Logics of Programs*, volume 193 of *Lecture Notes in Computer Science*, Springer-Verlag, 1985.

[BZ82] J.W. de Bakker and J.I. Zucker. Processes and the denotational semantics of concurrency. *Information and Control*, 54:70–120, 1982.

[EHJ93] L. Errington, C.L. Hankin, and T.P. Jensen. Reasoning about Gamma programs. In G.L. Burn, S.J. Gay, M.D. Ryan, editors, *Theory and Formal Methods 1993*, Workshop in Computing, Imperial College, London, Springer-Verlag, 1993.

[Esa74] L. Esakia. Topological Kripke models. *Soviet Mathematics Doklady*, 15:147–151, 1974.

[Fin73] K. Fine. Some connections between elementary and modal logic. In S. Kange, editor, *Proceedings of the Third Scandinavian Logic Symposium*, 1–15, North-Holland, Amsterdam, 1973.

[GH94] S.J. Gay and C.L. Hankin. A program logic for Gamma. In J.-M. Andreoli, C.L. Hankin and D. Le Meétayer, editors, *Coordination Programming, mechanisms, models and semantics*, 167–178, Imperial College Press, London, 1996.

[GHK+80] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, 1980.

[GR89] R.J. van Glabbeek and J.J.M.M. Rutten. The processes of de Bakker and Zucker represent bisimulation equivalence classes. In *J.W. de Bakker, 25 jaar Semantiek*, pages 243–246, CWI, Amsterdam, 1989.

[Gue81] I. Guessarian. *Algebraic Semantics*. Volume 99 of *Lecture Notes in Computer Science*, Springer-Verlag, 1981.

[HM85] M.C. Hennessy and R. Milner. Algebraic laws for non-determinism and concurrency. *Journal of the ACM*, 32(1):137–161, 1985.

[Joh82] P.T. Johnstone. *Stone Spaces*. Cambridge University Press, 1982.

[Man76] E.G. Manes. *Algebraic Theories*, volume 26 of *Graduate Texts in Mathematics*. Springer-Verlag, 1976.

[Mar79] G. Markowsky. Free completely distributive lattices. *Proceedings of the American Mathematical Society*, 74(2):227–228, 1979.

[Mil80] R. Milner. *A Calculus of Communicating Systems.* Volume 92 of *Lecture Notes in Computer Science*, Springer-Verlag, 1980.

[Par81] D.M. Park. Concurrency and automata on infinite sequences. In P. Deussen, editor, *Proceedings of the 5th GI Conference*, volume 104 of *Lecture Notes in Computer Science*, pages 167–183. Springer-Verlag, 1981.

[Plo81a] G.D. Plotkin. Post-graduate lecture notes in advanced domain theory (incorporating the 'Pisa notes'). Department of Computer Science, University of Edinburgh, 1981.

[Plo81b] G.D. Plotkin. A structural approach to operational semantics. Technical Report DAIMI FN-19, Computer Science Department, Aarhus University, 1981.

[Ran52] G. Raney. Completely distributive complete lattices. In *Proceedings of the American Mathematical Society*, volume 3(4), pages 677–680. Menasha, Wis., and Providence, R.I., 1952.

[Sc890] E. Schröder. *Vorlesunger über die Algebra der Logik*, volume I. B.G. Teubner, Leipzig, 1890. Republished in 1966 by Chelsea Publishing Co., New York.

[Sco70] D.S. Scott. Outline of a mathematical theory of computation. In *Proceedings 4th Annual Princeton Conference on Information Sciences and Systems*, pages 169–176, 1970.

[Sco82] D.S. Scott. Domains for denotational semantics. In M. Nielsen and E.M. Schmidt, editors, *9th International Colloquium on Automata, Languages and Programming; Aarhus, Denmark*, volume 140 of *Lecture Notes in Computer Science*, pages 577–613. Springer-Verlag, 1982.

[Smy83] M.B. Smyth. Power domains and predicate transformers: a topological view. In J. Diaz, editor, *Proceedings 10th International Colloquium on Automata, Languages and Programming; Barcelona, Spain*, volume 154 of *Lecture Notes in Computer Science*, pages 662–675. Springer-Verlag, 1983.

[Sto37] M.H. Stone. Topological representation of distributive lattices and Brouwerian logics. *Casopis Pro Potování Mathematiky*, 67:1–25, 1937.

[SS71] D.S. Scott and C. Strachey. Towards a mathematical semantics for computer languages. In *Proceedings of the Symposium on Computers and Automata*, volume 21 of *Microwave Research Institute Symposia series*, 1971.

[Vic89] S.J. Vickers. *Topology via Logic.* Volume 5 of *Cambridge Tracts in Theoretical Computer Science*, Cambridge University Press, 1989.

[Zha91] G.-Q. Zhang. *Logic of Domains.* Progress in Theoretical Computer Science, Birkhauser, 1991.