Logic in Computer Science

Introduction

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What is Logic?

- Logic from the Greek λογική
  - The tool for distinguishing between the true and the false (Averroes).
  - The science of the most general laws of truth (Gottlob Frege)
  - The study of the formal principles of reasoning (Webster dictionary)
  - The analysis and appraisal of arguments (Wikipedia)

If it was so, it might be, and if it were so, it would be; but as it isn’t, it ain’t. That’s logic!

Lewis Carol – via Tweedledeed in *Through the looking glass*
What is logic?

You drive from home to the University at 20 km/h and return by the same route at 30 km/h.

Discounting the time spent at the University, what was your average speed?
What is logic?

- **d** = distance Home - University
  
  **T** = time to get to University

  **t** = time get back

  \[
  d = 20T \quad \text{thus} \quad T = \frac{d}{20}
  \]

  \[
  d = 30t \quad \text{thus} \quad t = \frac{d}{30}
  \]

- **A** = average speed

  \[
  2d = A(T + t)
  \]

  \[
  2d = A\left(\frac{d}{20} + \frac{d}{30}\right) \quad \text{substitution}
  \]

  \[
  2d = A\left(\frac{3d}{60} + \frac{2d}{60}\right) \quad \text{calculation}
  \]

  \[
  2d = A\left(\frac{5d}{60}\right) \quad \text{calculation}
  \]

  \[
  A = \frac{120d}{5d} \quad \text{thus} \quad A = 24
  \]
What is logic?

A man is looking at a photograph of someone.

His friend asks who is she. The man replies, “I do not have brothers and sisters. But that girl’s father is my father's daughter.”

Who is portrayed in the photograph?
Is it logic?

- All men are intelligent
- Plato is a man
- Therefore, Plato is intelligent

- All men are intelligent
- Women are not men
- Therefore, women are not intelligent
Is it logic?

- All elms are trees.
- All oaks are trees.
- Therefore, all elms are oaks.
Trivium (three ways)

- Preparatory study at medieval university
  - **Grammar**, the art of combining symbols to express thoughts
  - **Logic**, the art of thinking
  - **Rhetoric**, the art of communicating thoughts

- to be studied before quadrivium
  - **Arithmetic**, geometry, music, and astronomy

- Simpler, thus … trivial!
The four ages of Logic

- **Symbolic logic** (500 b.C. – 19th century)

- **Algebraic logic** (mid – late 19th century)

- **Mathematical logic** (late 19th – mid 20th century)

- **Logic in computer science** (late 20th century – )
The origin of logic

- **Plato**: what is that can properly be called **true** or **false**?

- In formal debate **sophists** used rules to determine who had won an argument.

- Logic dealt with arguments in natural language used by humans.
Correct argument?

The guilty man

If Alice testifies against Bob in court, Bob will be found guilty.

Bob was found guilty.

Therefore, Alice must have testified against him.
Ambiguities and paradoxes

- All or one?
  - *Eric does not believe that Mary can pass any test.*

- Different meanings
  - *Students hate annoying professors*

- Paradox
  - *This sentence is a lie*
Symbolic logic

- **Aristotle’s syllogism (350 b.C.)**
  - Major premise: All men are mortal.
  - Minor premise: Socrates is a man.
  - Conclusion: Socrates is mortal.

- **Bacon (17th century)**
  - Not deduction, but induction should be used for drawing generalized conclusions from axioms or observations.

- **Kant (18th century)**
  - Logic is the one completed science, and Aristotelian logic included everything about logic there is to know.
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Algebraic Logic

- **Leibniz's dream** (1680): combining symbolic logic, mathematics and philosophy.
  - Formulate logic using a mathematical language
  - Ideas are composed from very few simple ones using few combinators like conjunction, negation, ...

- **Boole and De Morgan** (end 17th century)
  - Study of properties of conjunction, negation … of symbols in analogy to multiplication, subtraction of numbers
    - \( a(b + c) = (ab) + (ac) \)
    - \( a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \)
Algebraic Logic

- **Dodgson** (Lewis Carol) (1860)
  - Venn’s diagrams as a visualization of sets

- **Schröder's** “The algebra of Logic” (1890)
  - Logic is *calculating discipline, a scientific universal language that looks more like a sign language than like a sound language.*

- Logic become a useful tool to resolve many serious problems in mathematics
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Mathematical logic

- Mathematics become complex … and ambiguous
  - Cauchy (1820): infinite sum of continuous function is continuous
  - Abel (1826): counterexample!

- Frege (1879) proposes a new logic with quantifiers as a language for mathematics to resolve ambiguities
  - more powerful than Aristotelian (propositional) logic
  - Weierstrasse ($\varepsilon, \delta$) definition of limits and continuity
  - Peano axiomatization of natural numbers
  - Hilbert axiomatization of geometry
The golden age

- **Cantor** set theory, infinity of infinities, transfinite numbers
  - Mathematics is apart from physical phenomena
  - The absolute infinity nature of God is challenged

- **Hilbert’s program** (1920)
  - Finitary logic is the language of mathematics
  - **Completeness**: all true mathematical statements should be proved in this logic
  - **Consistency**: no contradiction can be obtained in this logic
  - **Decidability**: there should be an algorithm for deciding theorems.
The Russell paradox

- **Cantor** set theory is inconsistent (1901)
  - If $T = \{ x \mid x \notin T \}$ then $x \in T \iff x \notin T$

- **Russel** and **Whitehead** *Principia Mathematica* (1919):
  - Solution by using a hierarchy of sets (**type theory**)
    - Sets of a given type can contain sets of preceding types (those lower in the hierarchy), thus preventing loops.

- **Zermelo-Fraenkel** axiomatic set theory (1922)
  - Set theory as foundation of mathematics
Mathematics is more than sets

- **Continuum Hypothesis (CH)**
  - There is no set between the Integers and the Reals

- **Axiom of choice (AC)**
  - It is always possible to select an element from each non-empty set of an infinite collection of sets.

- **Goedel (1930) and Cohen (1960)**
  - Assuming or not either CH or AC does not give a contradiction in ZF set theory
  - There are mathematical statements that are independent from set theory
The end of Hilbert’s program

- **Goedel (1931)**
  - If a logic is strong enough to form statement about itself, then it cannot be *complete*
  - It is impossible to prove *consistency* of arithmetic within any formal theory of arithmetic

- **Church (1936), Turing (1937), Post (1946)**
  - There are problems that no algorithm will ever solve

- Logic is not the ultimate foundation of mathematics, simply another branch of it
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Computer Science started as Logic

- Outcome of Hilbert’s program
  - Formalization of algorithm (Church, Turing)
  - Stored-program computers (Turing machine, von Neumann)
  - Functional programming (Church $\lambda$-calculus)
  - Type theory (Russell, Whitehead, Church)
  - Recursive function theory (Goedel, Church, Turing, Kleene, Post)
  - Automata theory
Computer Science and Logic

- **Computer Science** = design and study of computational systems via formal languages

- **Logic** = formal language to study properties of mathematical structures

- Logic is the calculus of computer science
  - Logic has been more effective in computer science than it has been in mathematics
Logic and system verification

- Model checking

  Spec → Model checker → Yes!
  Model

  Spec

  Yes!

- Program verification

  Spec → System → Yes!
  System

  Yes!

  No
Logic and programming languages

- Type theory
- Program semantics
- Rewriting systems
- Functional programming
- Logic and constraint programming
Logic and databases

- Query languages (SQL)
  - Based on first-order logic

- Query evaluation
  - Based on relational algebra

- XML Data Type Definition
  - Based on tree automata

- XML queries
  - Based on tree transducers
Logic and ....

- Artificial intelligence
- Computational complexity
- Hardware circuits
- Security

...
Logic from computer science

- Logics on finite structures
- Fixpoint logics, modal and temporal logics
- Description logics, non-monotonic logic
- Connections between logic and automata
- Coinduction, coalgebra, bisimulation
Reference

Logic will get you from A to B. Imagination will take you everywhere.

Albert Einstein