1. [2 points] Give a proof by means of natural deduction of the following sequents:
   a) \( p \lor p \vdash p \).
   b) \( p \rightarrow q \vdash \neg q \rightarrow \neg p \).
   c) \( (p \lor q) \land r \vdash (p \land r) \lor (q \land r) \).
   d) \( \neg r \land (p \lor s) \vdash p \rightarrow (q \rightarrow \neg r) \).

2. [1 point] Use mathematical induction to prove that \( 2^0 + 2^1 + 2^2 + \ldots + 2^{n-1} = 2^n - 1 \) for all \( n \geq 1 \).

3. [1.5 points] Compute the conjunctive normal form of the following formulas:
   a) \( (p \land \neg q) \lor (p \land q) \).
   b) \( \neg(p \land \neg q) \land (q \land \neg p) \).
   c) \( (p \rightarrow q) \land (p \land \neg(r \land r \land q)) \).

Which formulas are valid? Explain your answer.

4. [1 point] Apply the marking algorithm to check if the following Horn formulas are satisfiable:
   a) \( (T \rightarrow q) \land ((p \land q) \rightarrow r) \land (q \rightarrow p) \).
   b) \( (T \rightarrow p) \land ((p \land q) \rightarrow r) \land (p \rightarrow q) \land ((r \land p) \rightarrow q) \).
   c) \( (T \rightarrow p) \land (p \rightarrow q) \land ((p \land q) \rightarrow r) \land (q \rightarrow \bot) \land (T \rightarrow r) \).

5. [1.5 points] Write predicate formulas expressing the following statements (use no more than one predicate symbol but as many function symbols as you want):
   a) There is an integer that is even and there is an integer that is odd.
   b) Each integer is even.
   c) No integer is prime and even.

Explain your answer and do not forget to mention your choice of predicate and function symbols (each together with its arity).

6. [2 points] Show the validity by means of natural deduction of the following sequents:
   a) \( \forall x \ P(x) \vdash \neg P(a) \lor P(b) \).
   b) \( a = b \land \neg P(a,b) \vdash \neg \forall x \ P(x,x) \).
   c) \( \neg \forall x (x = x \lor \neg (x = x)) \).
   d) \( \forall \exists x \neg (x = x) \).

7. [1 point] Consider the predicate formula \( \forall x \exists y \ (x < y \rightarrow x + 1 = y) \). Find two models, one which makes the formula true and the other which makes the formula false.

The final score is given by the sum of the points obtained.