Question 1: 

[1.5 points]

a) Give a regular expression for the language \( L_1 = \{ w \in \Sigma^* \mid \text{w does not begin with 11} \} \) and one for the language \( L_2 = \{ w \in \Sigma^* \mid \text{w ends with 00} \} \).

A regular expression for the language \( L_1 \) is \( \Lambda + 1 + (0 + 10)(0 + 1)^* \). One for the language \( L_2 \) is \( (0+1)^*00 \).

b) Give regular expressions for the union, concatenation and intersection of the two languages \( L_1 \) and \( L_2 \) above.

The language denoted by \( \Lambda + 1 + (0 + 10)(0 + 1)^* + (0+1)^*00 \) is \( L_1 \cup L_2 \); whereas the language denoted by the regular expression \( \Lambda + 1 + (0 + 10)(0 + 1)^*(0+1)^*00 \) is \( L_1 \cdot L_2 \). The intersection of \( L_1 \) with \( L_2 \) is the language of all strings \( w \in \Sigma^* \) that do not begin with 11 and end with 00. This is the same as the language \( L_1 \cdot \{00\} \), therefore a regular expression for \( L_1 \cap L_2 \) is \( (\Lambda + 1 + (0 + 10)(0 + 1)^*)00 = 00 + 100 +(0+10)(0+1)^*00 \).

c) Explain why for all regular expression \( R \), the language of \( R + \emptyset \) is the same of that of \( R \emptyset \), but the language of \( R + \Lambda \) is never equal to that of \( R \emptyset \).

For every regular expression \( R \) we have

\[
L(R + \emptyset) = L(R) \cup L(\emptyset) = L(R) \cup \emptyset = L(R) = L(R)\{\Lambda\} = L(R)\Lambda = L(R\Lambda).
\]

Further, \( L(R\emptyset) = \emptyset \), while \( \Lambda \in L(R + \Lambda) = L(R) \cup \{\Lambda\}. \) Thus the two languages are different for every regular expression \( R \).

Question 2: 

[1.5 points]

a) Give a non-deterministic finite automaton with at most three states accepting the language of the regular expression \( \Lambda + (a + aa^*b)b^* \).

![Diagram of non-deterministic finite automaton]

b) Using the powerset construction, find a deterministic finite automaton that recognizes the same language as the non-deterministic finite automaton in the above item a).

![Diagram of deterministic finite automaton]
c) Suppose \( M = (Q, \Sigma, q_0, A, \delta) \) is a non-deterministic finite automaton recognizing the language \( L \).
Let \( M_1 \) be the automaton obtained from \( M \) by adding a \( \Lambda \)-transition from each state in \( A \) to \( q_0 \).

Describe the language \( L(M_1) \) in terms of \( L \).

A string \( w \) is accepted by \( M_1 \) if it can be decomposed in \( w_1 \cdots w_n \) such that each \( w_i \) is accepted by \( M \). This means that \( L(M_1) = L(M)^+ \). Note that since \( q_0 \) needs not to be an element of \( A \), the string \( \Lambda \) is, in general, not accepted by \( M_1 \), explaining why \( L(M_1) \neq L(M)^* \).

**Question 3:**  
[1.5 points]

Give a counterexample proving that each of the following statements on languages over the alphabet \( \{a,b\} \) are false.

a) If \( L_1 \subseteq L_2 \) and \( L_1 \) is not regular, then also \( L_2 \) is not regular.

Take \( L_1 = \{a^n b^n | n \geq 0 \} \) and \( L_2 = \{a,b\}^* \). Then \( L_1 \) is non-regular but \( L_2 \) is.

b) If \( L_1 \) is regular, \( L_2 \) is not regular and \( L_1 \cap L_2 \) is regular, then \( L_1 \cup L_2 \) is regular.

Take \( L_1 = \emptyset \) and \( L_2 = \{a^n b^n | n \geq 0 \} \). Then \( L_1 \) is regular, \( L_2 \) is not, their intersection is \( L_1 \) and therefore regular, but their union is \( L_2 \) and thus a non-regular language.

c) If \( L_1 \) is regular, \( L_2 \) is not regular and \( L_1 \cup L_2 \) is regular, then \( L_1 \cap L_2 \) is regular.

Take \( L_1 = \{a,b\}^* \) and \( L_2 = \{a^n b^n | n \geq 0 \} \). Then \( L_1 \) is regular, \( L_2 \) is not, their union is \( L_1 \) and therefore regular, but their intersection is \( L_2 \) and thus a non-regular language.

**Question 4:**  
[1 point]

Let \( G = <\{S\}, \{a,b\}, P, S> \) be a context free grammar with productions \( P \) given by

\[
S \rightarrow aSSb \mid aS \mid aSb \mid ab \mid \Lambda
\]

By eliminating \( \Lambda \)-productions we obtain the grammar \( S \rightarrow aSSb \mid aS \mid aSb \mid ab \mid a \).

**Question 5:**  
[2 points]

a) Give a context-free grammar for the language of all strings over \( \{0,1\} \) such that each block of 0’s is followed by at least as many 1’s.

\[
S \rightarrow AS \mid \Lambda \quad A \rightarrow 0A1 \mid A1 \mid 01
\]

b) Show a leftmost derivation in your grammar of the strings 0100111.

\[
S \Rightarrow AS \Rightarrow 01S \Rightarrow 01AS \Rightarrow 010A1S \Rightarrow 010A11S \Rightarrow 0100111
\]

c) Show a rightmost derivation in your grammar of the strings 0100111.

\[
S \Rightarrow AS \Rightarrow AAS \Rightarrow AA \Rightarrow A0A1 \Rightarrow A0A11 \Rightarrow A00111 \Rightarrow 0100111
\]

**Question 6:**  
[2.5 points]

Let \( L_1 = \{w \in \{a,b\}^* | w \text{ contains at least one } b \} \) and \( L_2 = \{w_1w_2 \in \{a,b\}^* | |w_1| = |w_2| \} \).

a) Give a deterministic finite automaton recognizing the language \( L_1 \).

b) Give a pushdown automaton recognizing the language \( L_2 \).
c) Use your two automata above to construct a pushdown automaton for the language $L_1 \cap L_2$.

The transitions in red are not strictly necessary.

The final score is given by the sum of the points obtained.