Inleiding Fundamentele Informatica 2
The Method of Brzozowski and McCluskey

Addition to 4.3 Kleene's Theorem
in John Martin: Introduction to Languages and the Theory of Computation

In addition to section 4.3, we consider here a different method to find an equivalent regular expression for a finite automaton. This method is still based on the $L(p, q, k)$ construction of the proof of Theorem 4.5, but is easier to apply since the amount of bookkeeping is reduced.

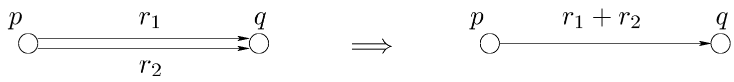
At the intermediate stages of the construction we will be dealing with generalized finite automata which may have arrows labelled by regular expressions. A regular expression labelling an arrow from state p to state q represents a set of input strings which may lead the automaton from p to q. Observe that every symbol from the input alphabet as well as Λ can be considered as a regular expression. Consequently, all FA, NFA, and NFA-Λ can be seen as generalized finite automata!

Any finite automaton or NFA or NFA-Λ can be modified in such a way that the resulting NFA-Λ has an initial state without incoming arcs and a single final (accepting) state, and no outgoing arcs from that accepting state, and still accepts the same language (see Exercise 4.26).

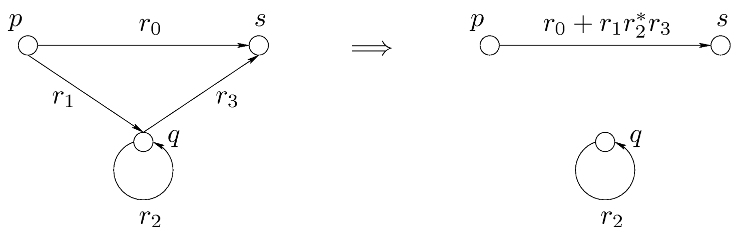
Thus from here on, we can assume that the input to our construction is an NFA-Λ $M = (Q, \Sigma, q_0, \{q_f\}, \delta)$ with no incoming arcs for its initial state q_0, with a single final state q_f, and no arrows leading out of q_f.
We describe a method to eliminate states from M and change the labels of the arrows in such a way that in the end only q_0 and q_f remain. Moreover, if $L(M) \neq \emptyset$, then there will be one arc left arc left, leading from q_0 to q_f and labelled with a regular expression r such that $L(M) = ||r||$ the regular language defined by r.

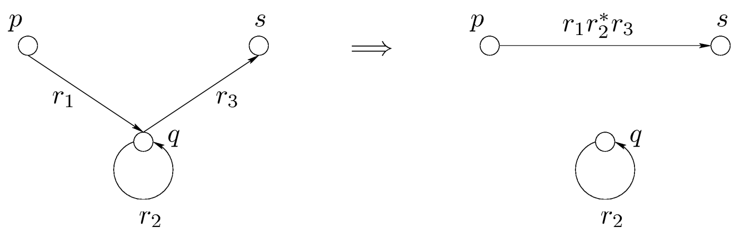
NFA-Λ M $\Longrightarrow$ $||r|| = L(M)$

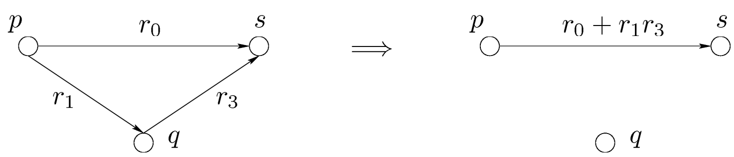
First we replace all parallel edges as follows:

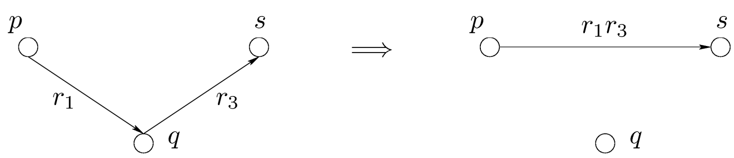
0) 

Next, if q is a state such that $q \neq q_0$ and $q \neq q_f$,
then for all states p and s such that $p \neq q$ and $s \neq q$ (we allow $p = s$),
and such that there is an arrow from p to q and one from q to s,
the automaton is modified as indicated below (we distinguish 4 cases),
after which q is deleted.

1) 

2) 

3) 

4) 

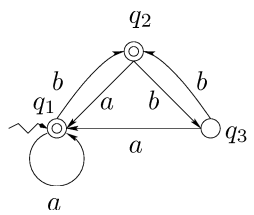
Observe that the new regular expression labelling the arrow from p to s describes the same regular language as the original regular expression leading

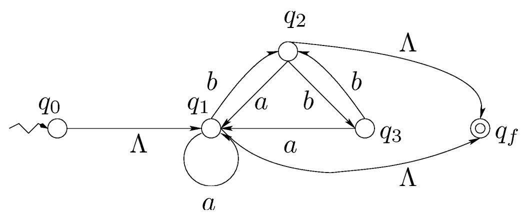
directly from p to s combined (+) with the regular expression leading from p to s via q.
Note furthermore that the cases 2,3,4 are each a special case of 1 (with $r_0 = \emptyset$ and/or $r_2 = \emptyset$).

In this way we can systematically remove all states different from q_0 and from q_f. If at this point, there is no arrow leading from q_0 to q_f then in M already q_f is not reachable from q_0 and $L(M) = \emptyset$. Otherwise, there is an arrow leading from q_0 to q_f labelled with a regular expression r and we have that $||r|| = L(M)$, because the regular expressions introduced describe the words corresponding to the original path labels.

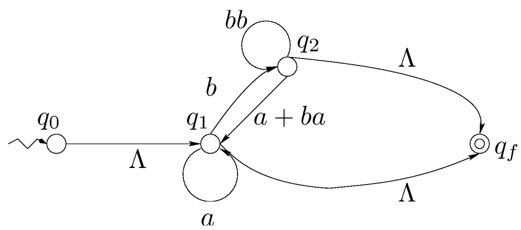
Finally, we note that the order in which states are removed is not fixed, but may be arbitrarily chosen. In general, different orders lead to different (but equivalent) expressions. Some orders may be more efficient than others.

EXAMPLE 4.10
As an example, we apply the method to the FA M of Example 4.10 (see Figure 4.18). First we bring M in the desired form with initial state q_0 without incoming arcs and the single final state q_f with no outgoing arcs.

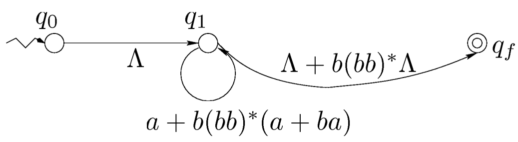




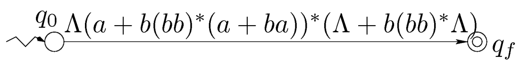
Next, we remove state q_3. Observe that the pairs q_2,q_1 and q_2,q_2 are the only pairs of states of which the first one has an outgoing arrow leading to q_3 and the second one has an incoming arrow from q_3.



Then we remove state q_2. We only have to consider the pairs q_1,q_1 and q_1,q_f.



Finally we remove state q_1; only q_0 and q_f need to be considered.



We are done, a regular expression for $L(M)$ is as indicated.
It is easy to see that this expression can immediately be simplified to the equivalent: $(a + b(bb)^*(a + ba))^*(\Lambda + b(bb)^*)$.

By the way, we have chosen first q_3, then q_2, and finally q_1, because in that way we minimized at each step the number of pairs to be considered (see for yourself what happens when the states are removed in the order q_1, q_2, q_3).

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