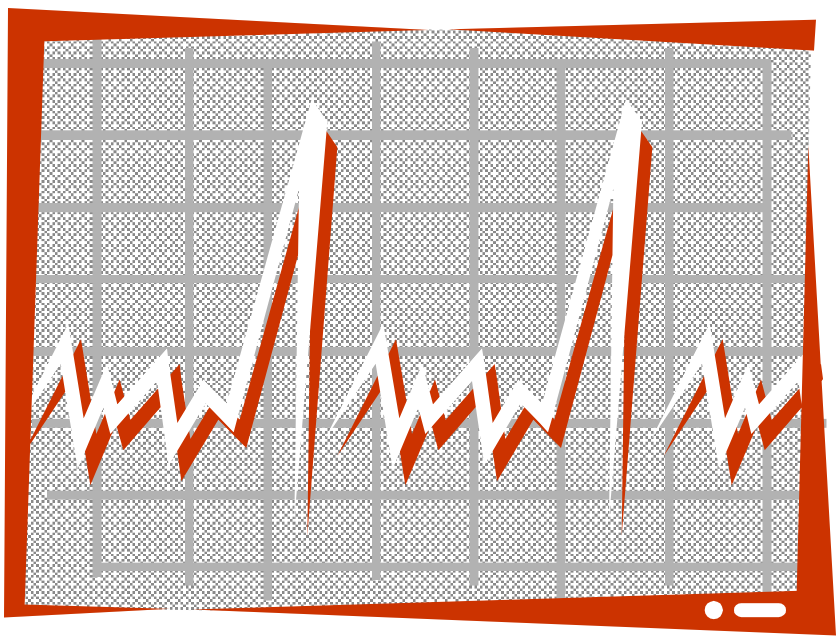
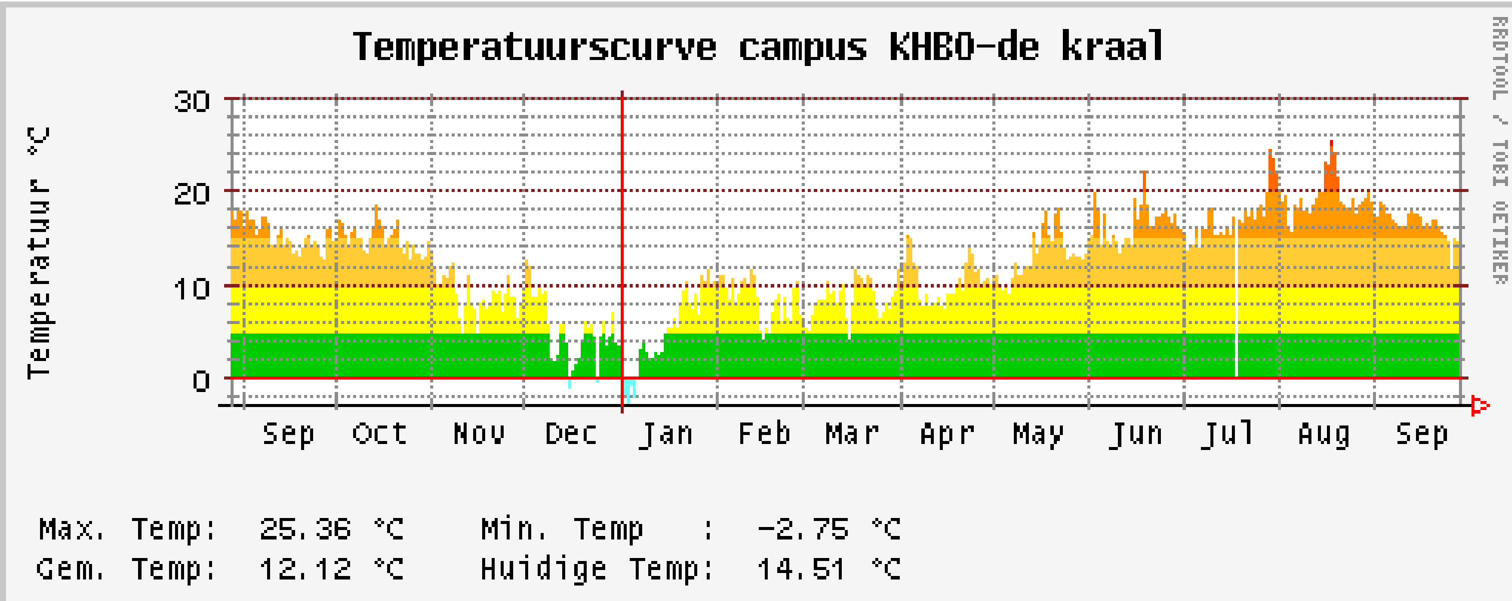
3

Functies



Ch.3 'Functions ~~and Algorithms~~'

temperatuur



$d \mapsto T_d \quad d \in \{ 28.8.01, \ldots 30.9.02 \}$

$T_d \in \mathbb{R}$

oneindige reeks

0102010301020104010201030102010501020103…

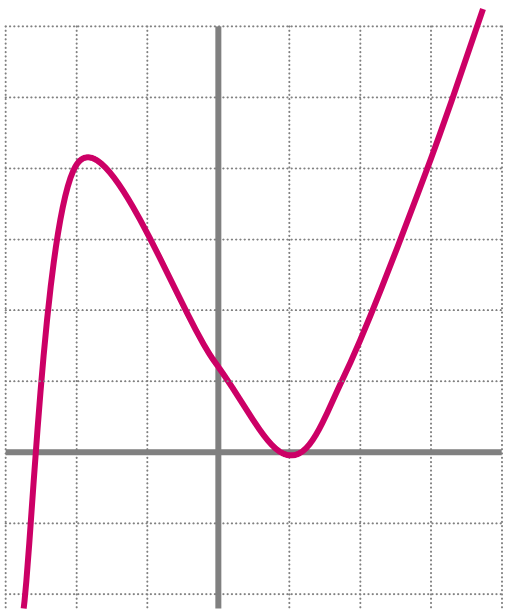
1	1001
10	1010
11	1011
100	1100
101	1101
110	1110
111	1111
1000	10000

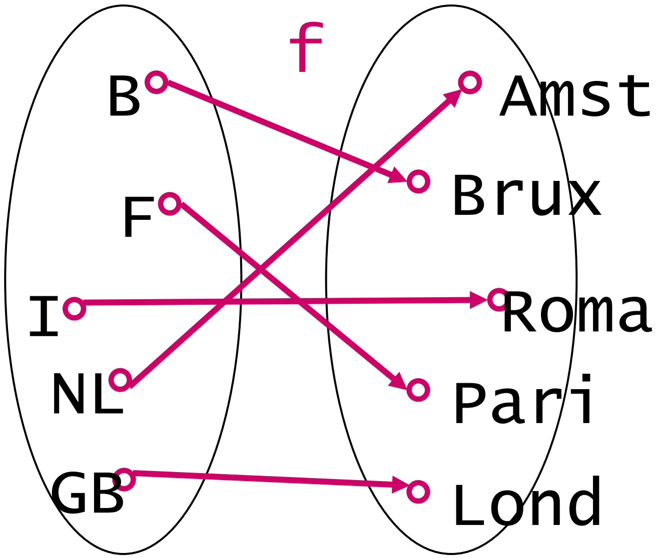
$n \mapsto \mathtt{tail}(n)$

$0 \mapsto 0,\ 1 \mapsto 1,\ 2 \mapsto 0,\ 3 \mapsto 2,$
$4 \mapsto 0,\ 5 \mapsto 1,\ 6 \mapsto 0,\ 7 \mapsto 3,$
$8 \mapsto 0,\ 9 \mapsto 1,\ 10 \mapsto 0,\ \ldots$

informeel

Een *functie* van A naar B is een voorschrift dat aan ieder element van A één element van B toevoegt.





0,1,0,2,0,1,0,
3,0,1,0,2,0,1,
0,4, …

	Am	Br	Lo	Pa	Ro
B		x			
F				x	
GB			x		
I					x
NL	x				

$$f: A \to B$$
$$f(x) = y$$
$$x \mapsto y$$

grafiek van een functie

functie $f: A \to B$. de *grafiek* van f is verzameling
$\{ (x,f(x)) \mid x \in A \}$ in $A \times B$.

$f(x) = 2(x+1)$ vs. $g(x) = 2x+2$

informeel ‘voorschrift’
⇨ formeel ‘relatie’

relatie →
binaire relatie
→ functie

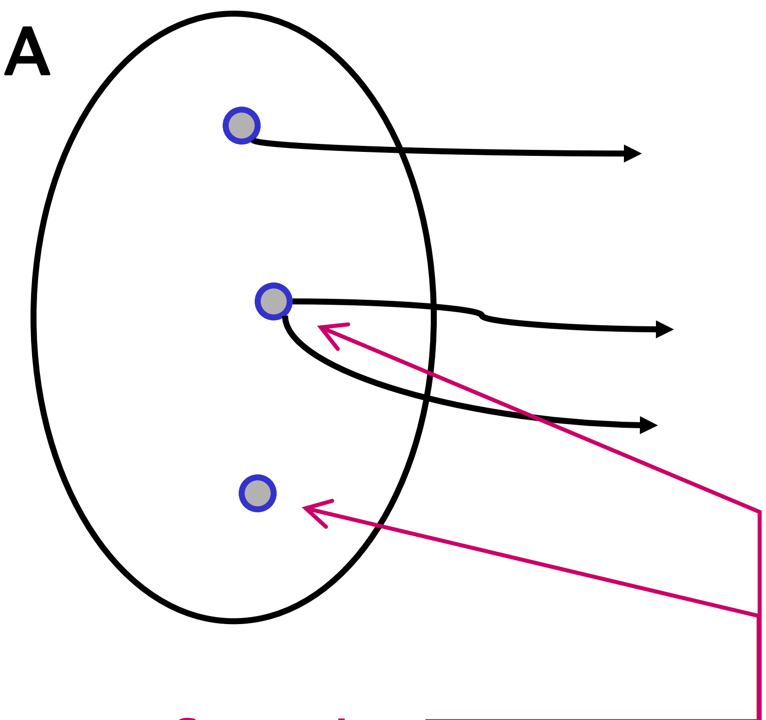
functies zijn gelijk als hun grafieken gelijk zijn, dwz. hun ‘uitkomsten’

$f = g \iff f(a)=g(a)$ voor alle $a \in A$

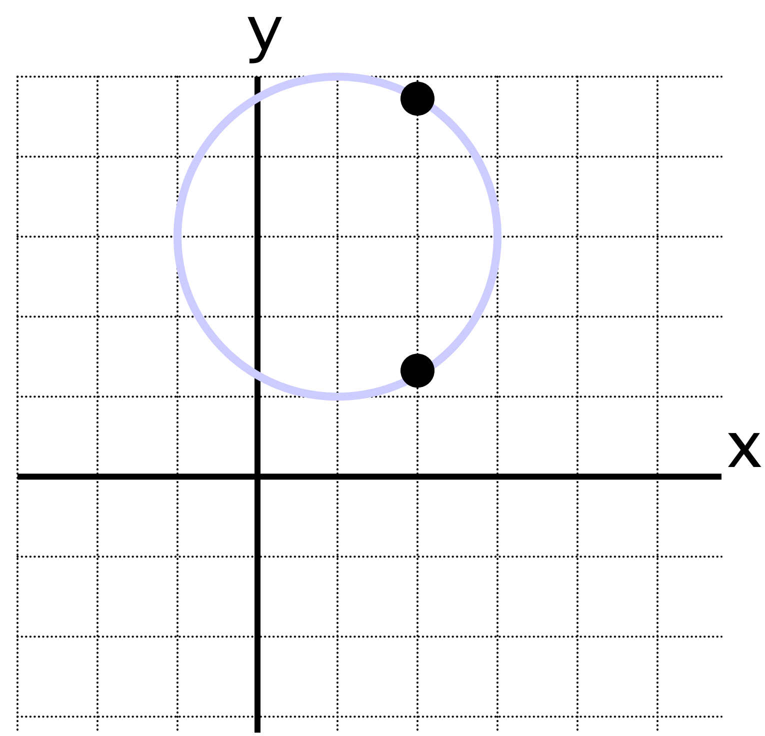
relatie $R \subseteq A \times B$ heet een *functie* van A naar B als voor iedere $x \in A$ er precies één paar xRy bestaat.

totaal & functioneel

$f: A \rightarrow B \quad y = f(x)$



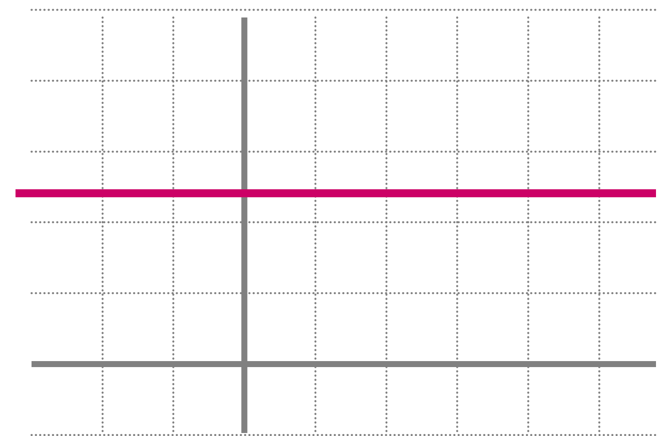
geen functie

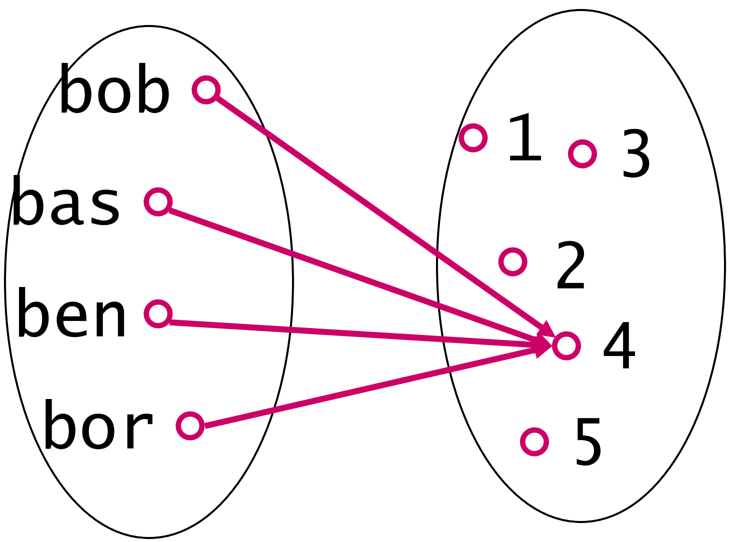


constante functie

functie $f: A \to B$. $f(x) = f(y)$ voor alle $x,y \in A$

	a	s	d	f	g
1	0	1	0	0	0
2	0	1	0	0	0
3	0	1	0	0	0
4	0	1	0	0	0
5	0	1	0	0	0

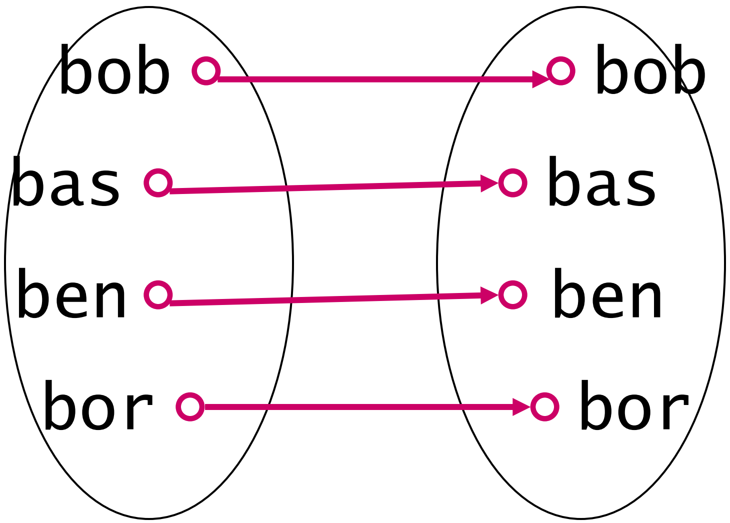


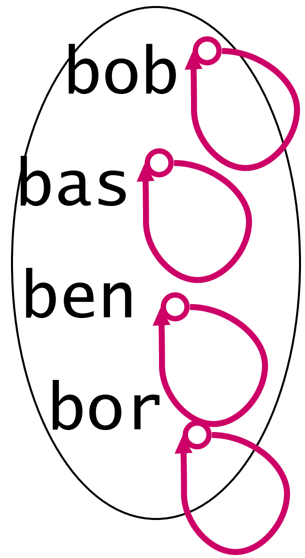


4,4,4,4,4,4,4,…

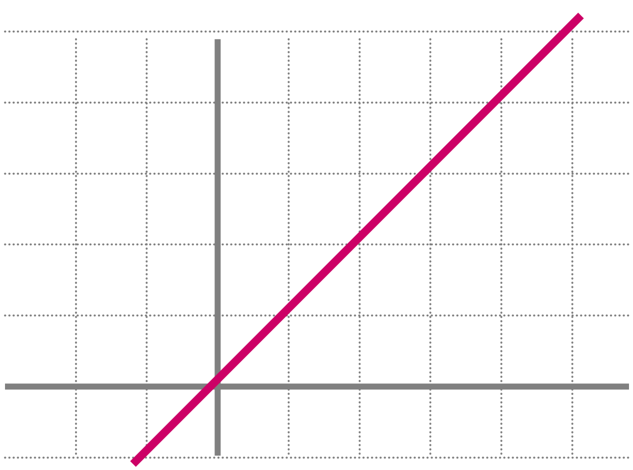
identiteit

functie $f: A \to A$ met $f(x) = x$ voor alle $x \in A$
identieke functie, *identiteit* op A.   notatie 1_A id_A



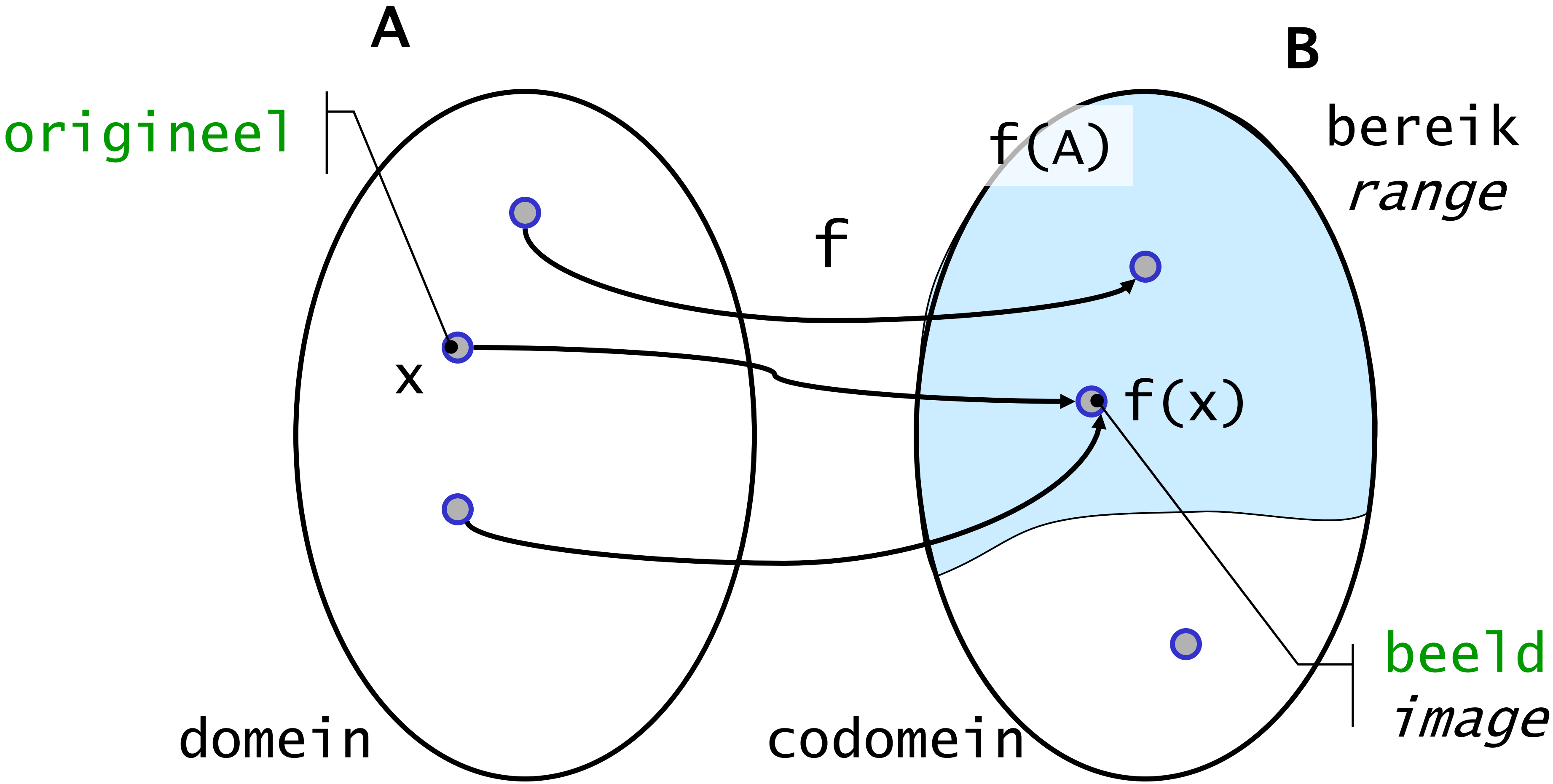


	1	2	3	4	5
1	1	0	0	0	0
2	0	1	0	0	0
3	0	0	1	0	0
4	0	0	0	1	0
5	0	0	0	0	1



0,1,2,3,4,5,6, …

f:A→B


A
B
origineel
f(A)
bereik
range
f
x
f(x)
beeld
image
domein
codomein

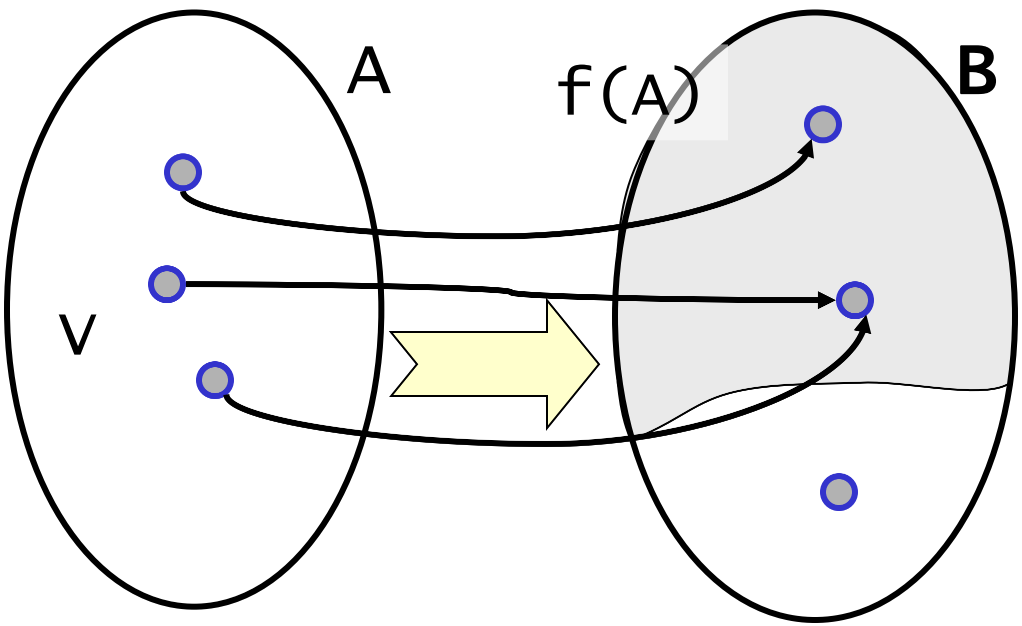
bereik

f: A → B

origineel ⇨ beeld

element

verzameling

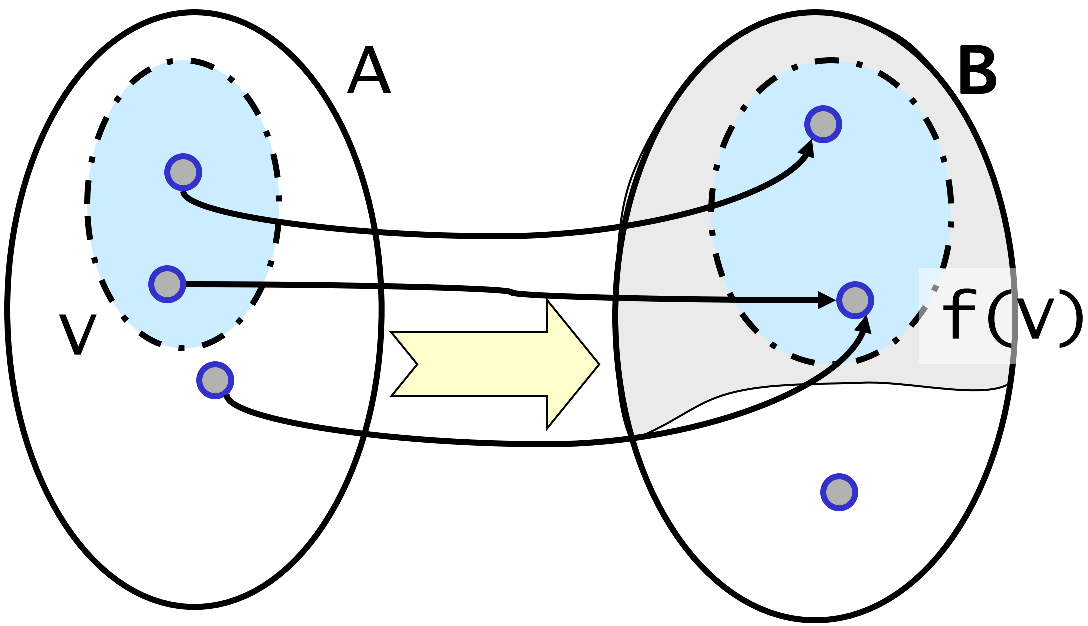


origineel & beeld

beeld $V \subseteq A$ $\qquad$ $f: A \to B$

$\{\ y \in B \mid y = f(x)$ voor zekere $x \in V\ \}$

$f(V) = \{\ f(x) \mid x \in V\ \}$



$V \subseteq A$

$\Rightarrow$

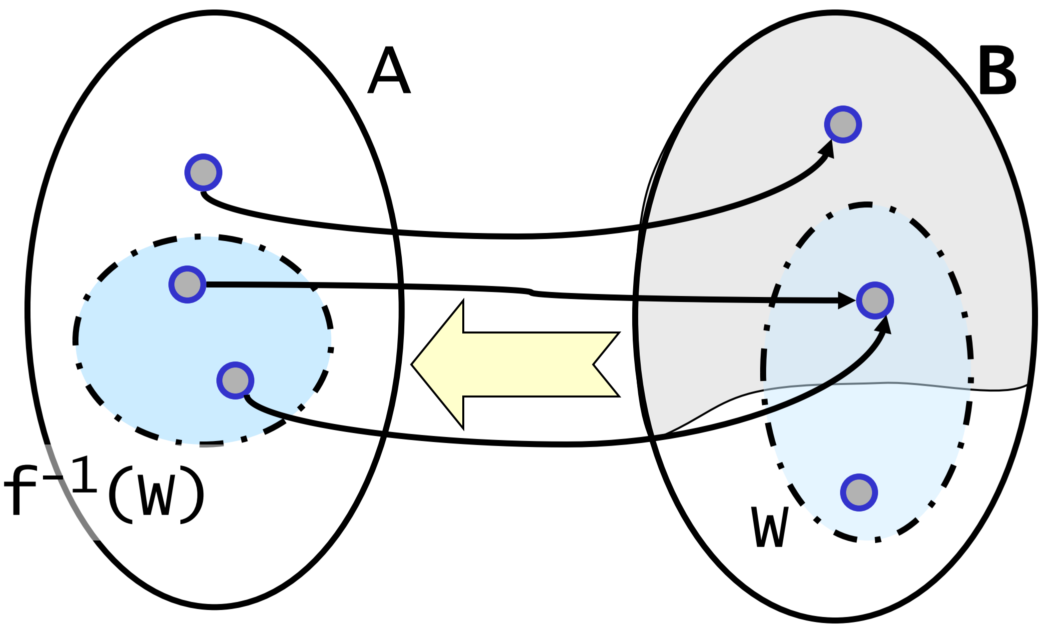
$f(v) \subseteq B$

<u>beeld</u>

origineel $W \subseteq B$ $f: A \to B$

$\{ x \in A \mid f(x) = y$ voor zekere $y \in W \}$

$f^{-1}(W) = \{ x \in A \mid f(x) \in W \}$

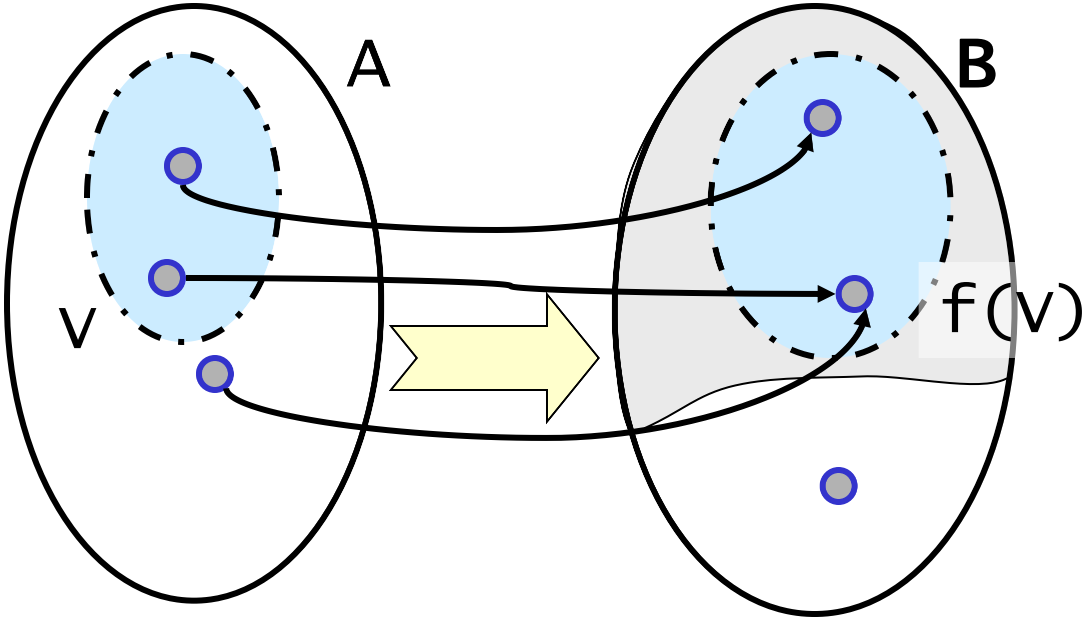


$W \subseteq B$

$\Rightarrow$

$f^{-1}(W) \subseteq A$

volledig origineel

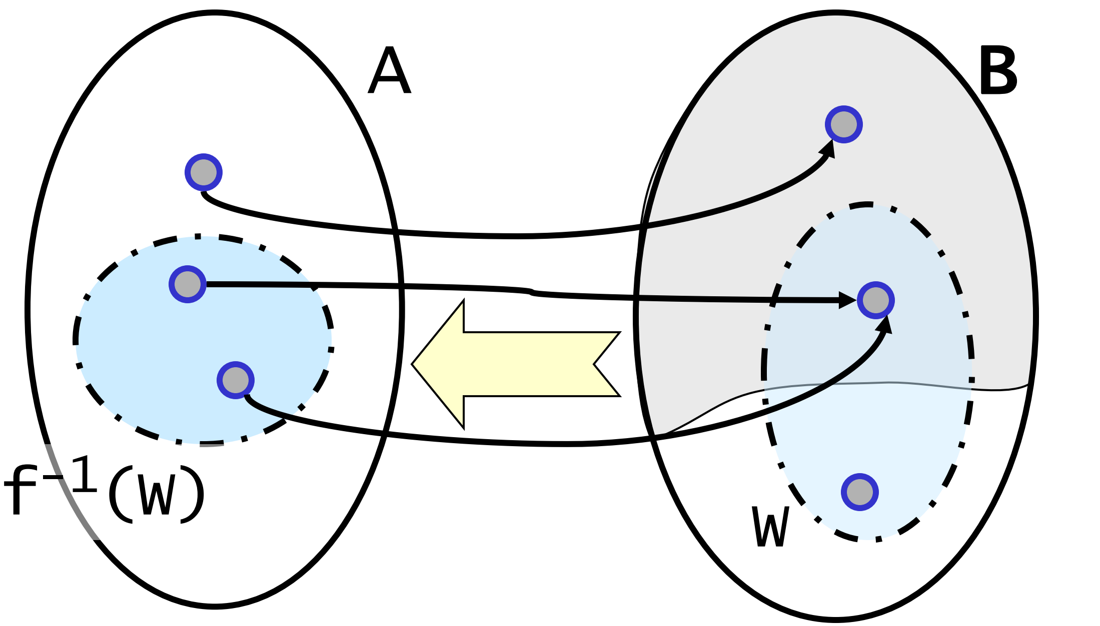


$V \subseteq A$

$\Rightarrow$

$f(v) \subseteq B$

beeld

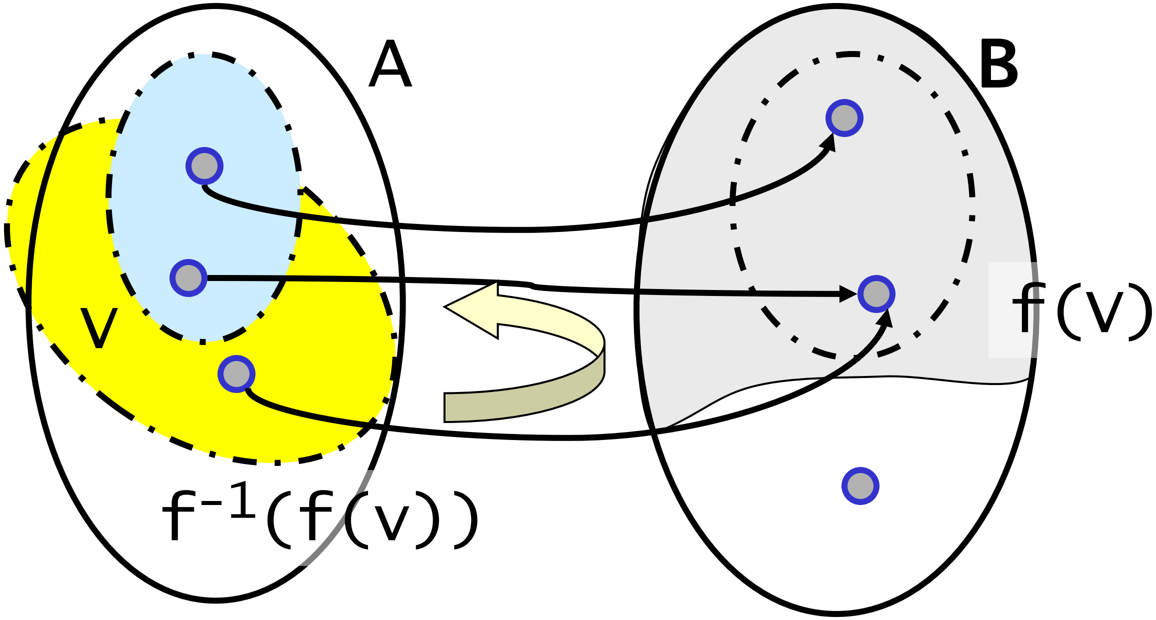


$W \subseteq B$

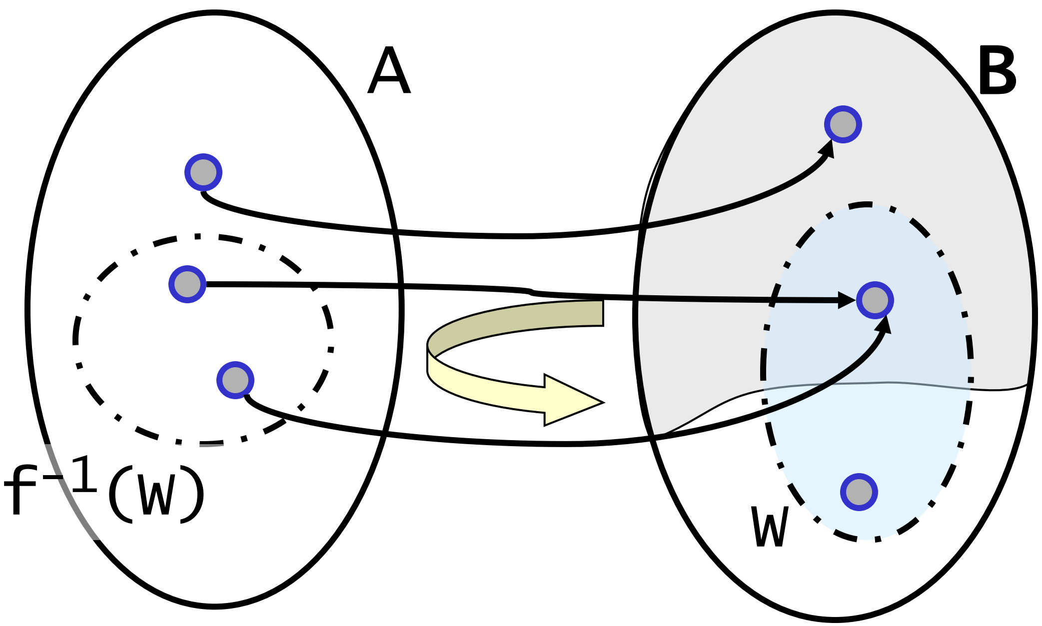
$\Rightarrow$

$f^{-1}(W) \subseteq A$

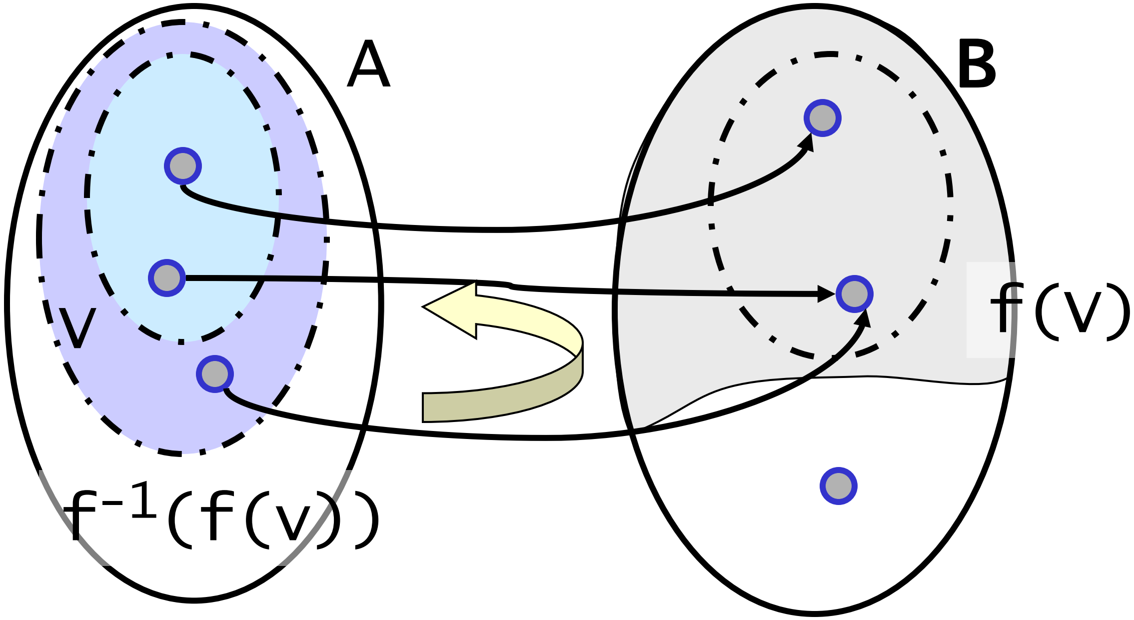
volledig origineel



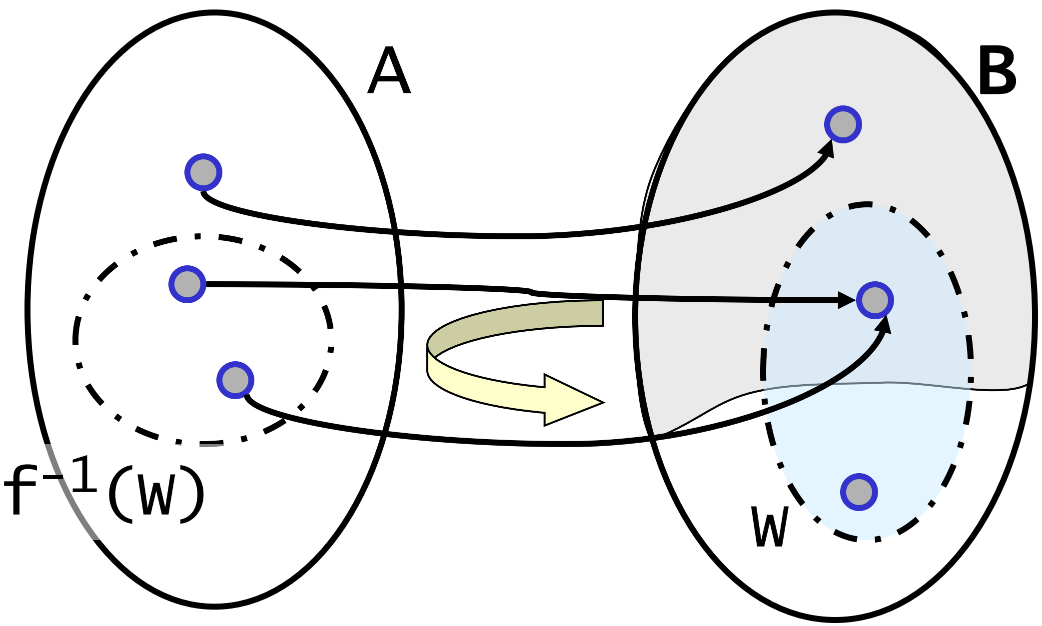
$$V \subseteq A$$
$$\Rightarrow$$
$$V \ ? \ f^{-1}(f(V))$$



$$W \subseteq B$$
$$\Rightarrow$$
$$f(f^{-1}(W)) \ ? \ W$$



$$V \subseteq A$$
$$\Rightarrow$$
$$V \subseteq f^{-1}(f(v))$$

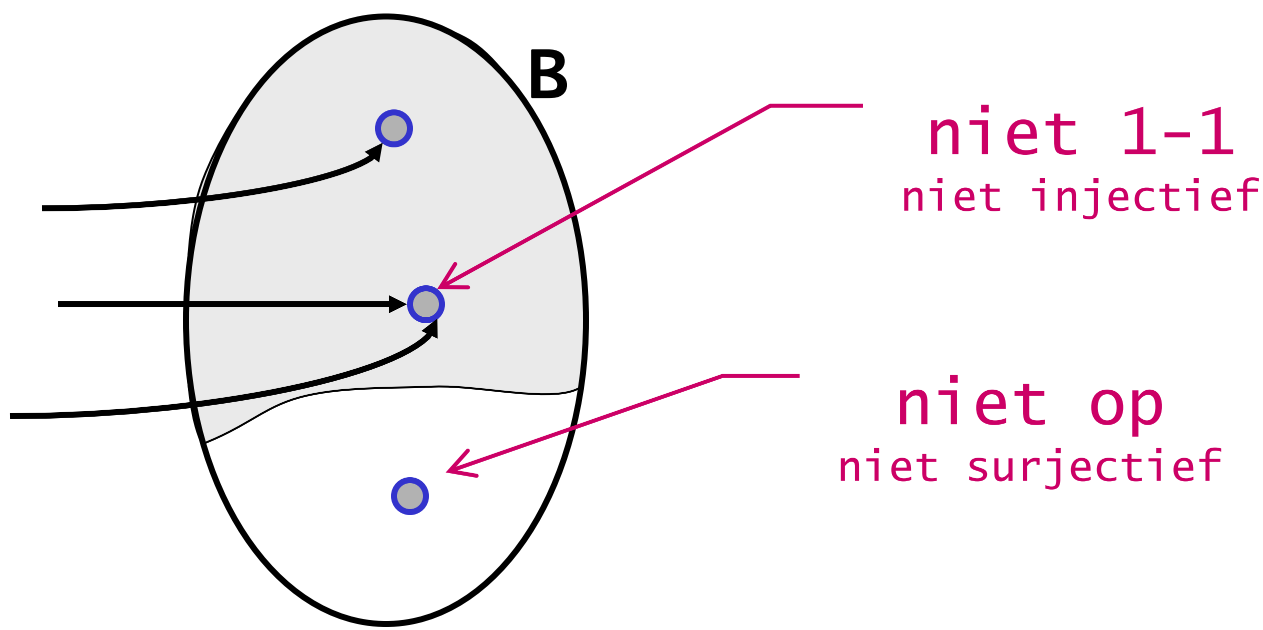


$$W \subseteq B$$
$$\Rightarrow$$
$$f(f^{-1}(W)) \subseteq W$$

$$f(f^{-1}(W)) = W \cap f(A)$$

Een functie $f: A \to B$ heet *surjectief* (of '*op*') als $f(A) = B$, met andere woorden, als er voor iedere $y \in B$ een origineel $x \in A$ bestaat.

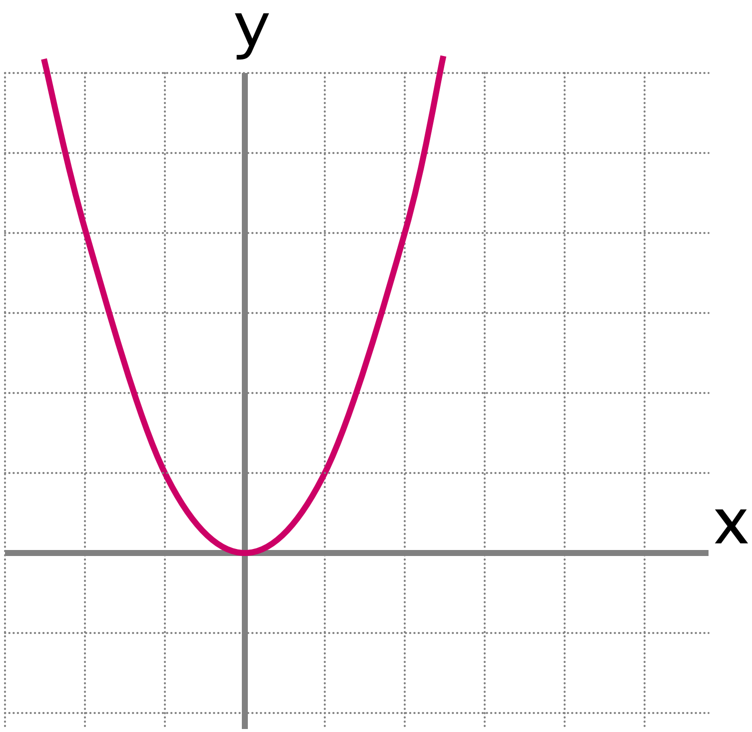
Een functie $f: A \to B$ heet *injectief* (of *een-eenduidig* of *een-op-een*) als voor alle $x, y \in A$ geldt dat uit $f(x) = f(y)$ volgt dat $x = y$.

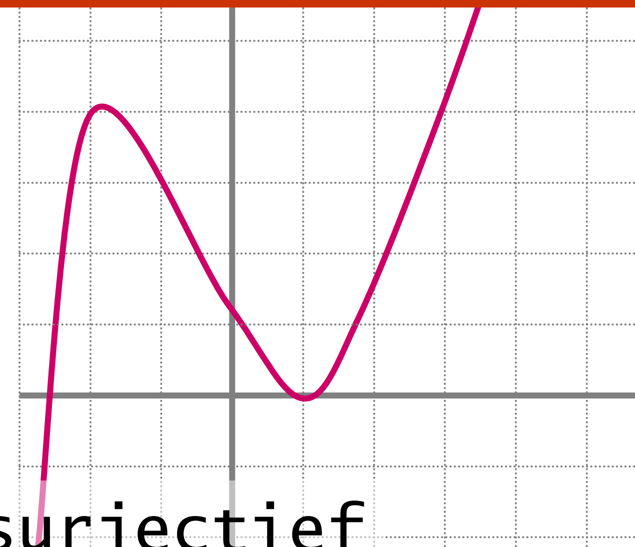


voorbeeld

$f: \mathbb{R} \to \mathbb{R}$

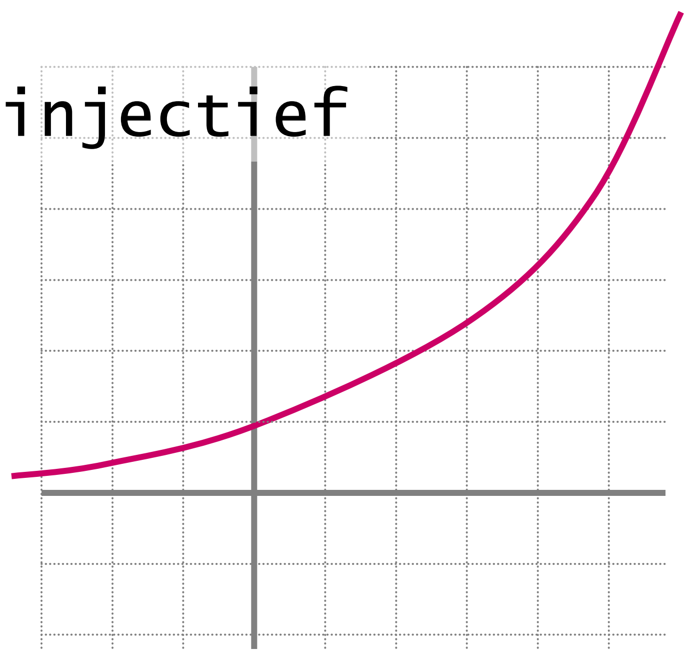
$f(x) = x^2$



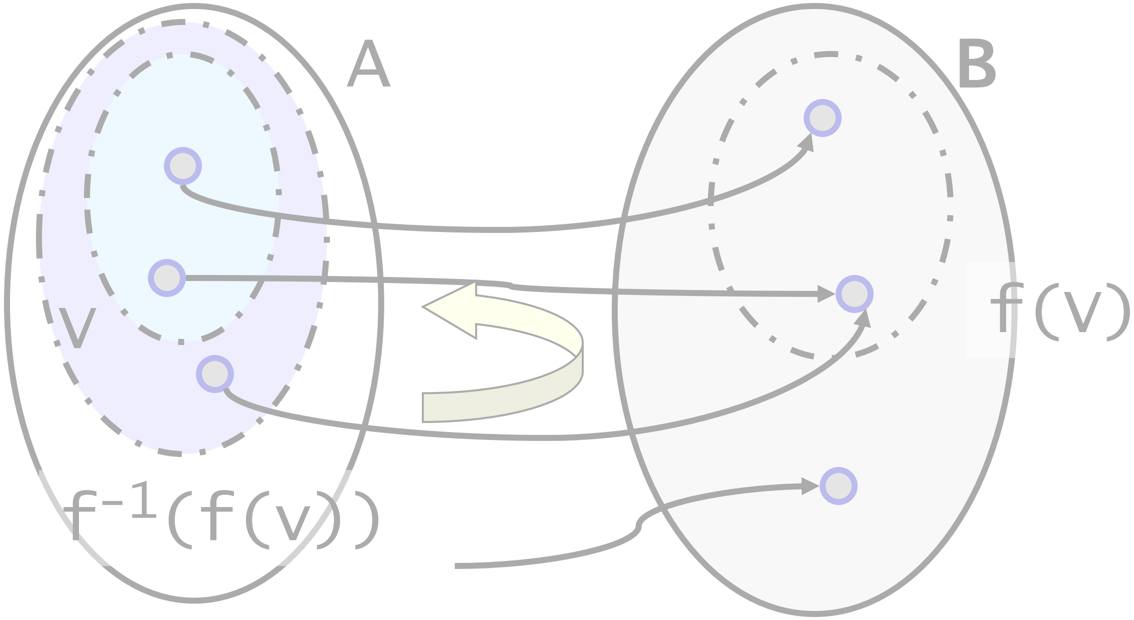


surjectief

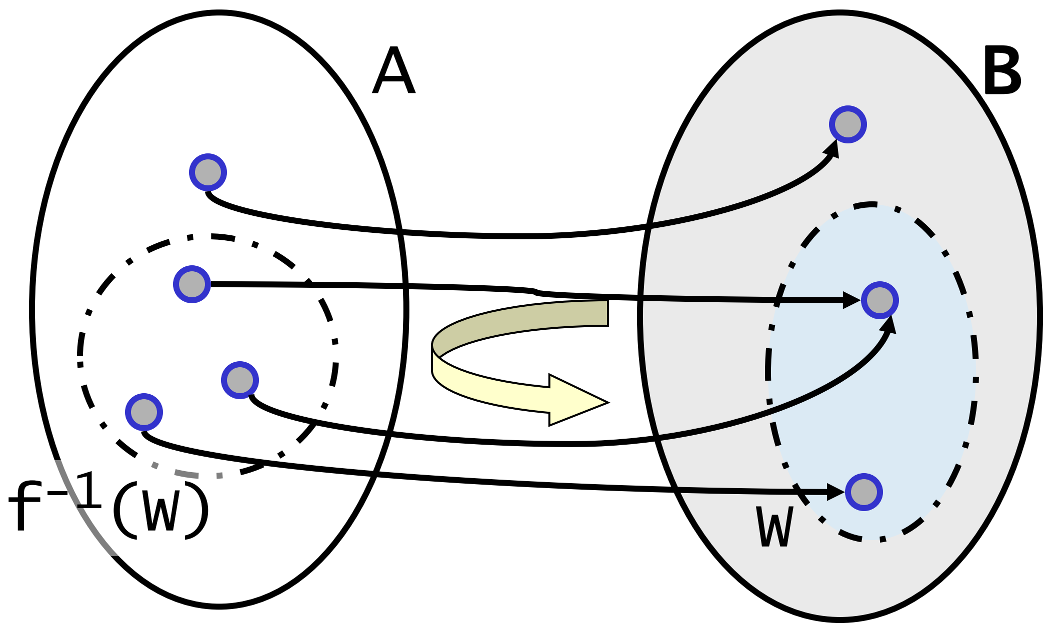
injectief



f surjectief ‘op’

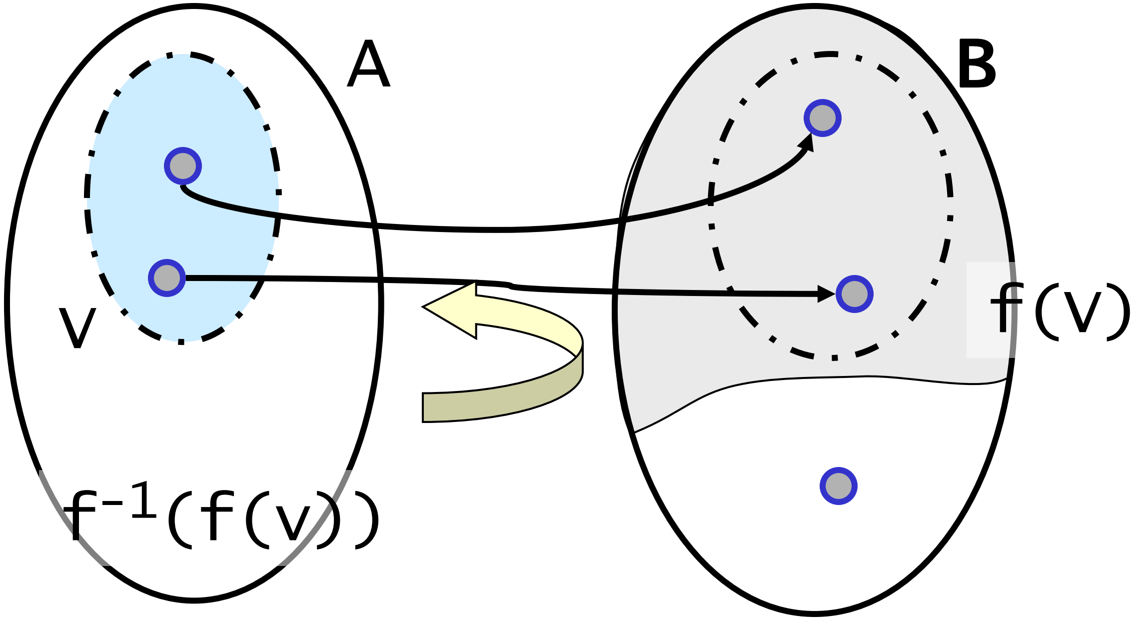


$$V \subseteq A$$
$$\Rightarrow$$
$$V \subseteq f^{-1}(f(V))$$

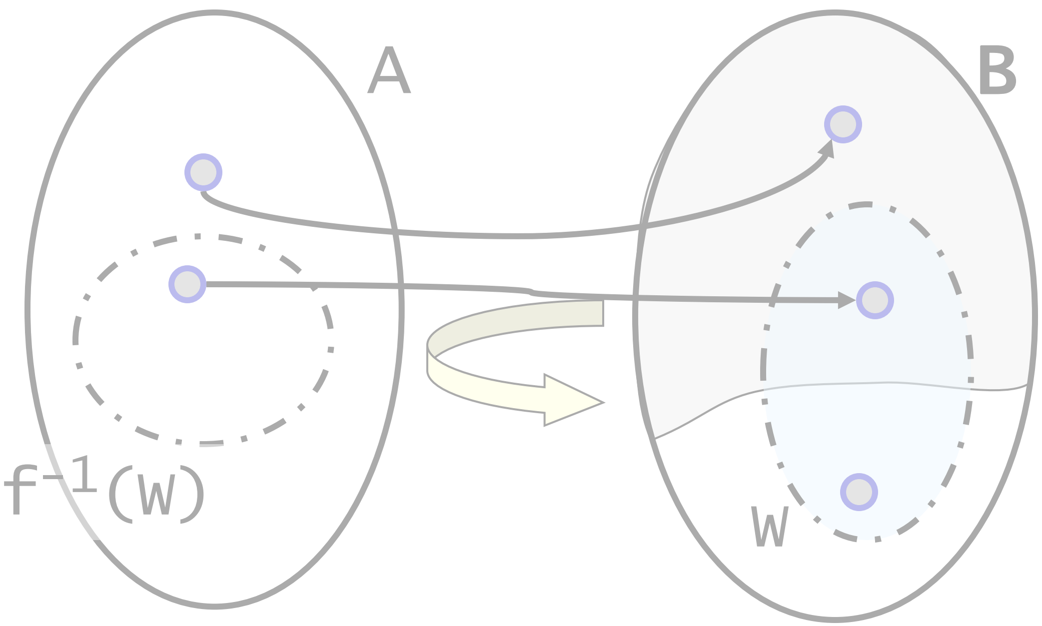


$$W \subseteq B$$
$$\Rightarrow$$
$$f(f^{-1}(W)) = W$$

f injectief ‘1-1’



$$V \subseteq A \Rightarrow V = f^{-1}(f(V))$$



$$W \subseteq B \Rightarrow f(f^{-1}(W)) \subseteq W$$

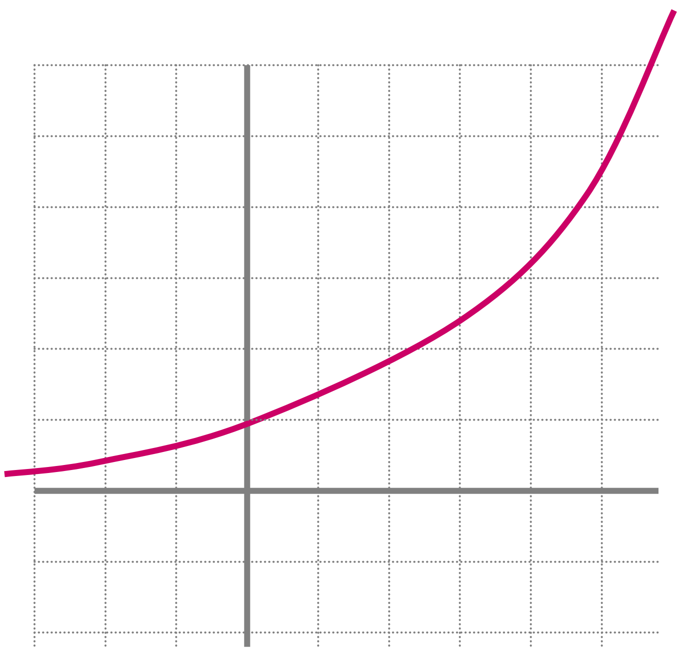
bijectie

Een functie $f: A \to B$ heet *bijectief* als f zowel surjectief als injectief is.

$V \subseteq A, \quad W \subseteq B$

$V = f^{-1}(f(V))$

$W = f(f^{-1}(W))$

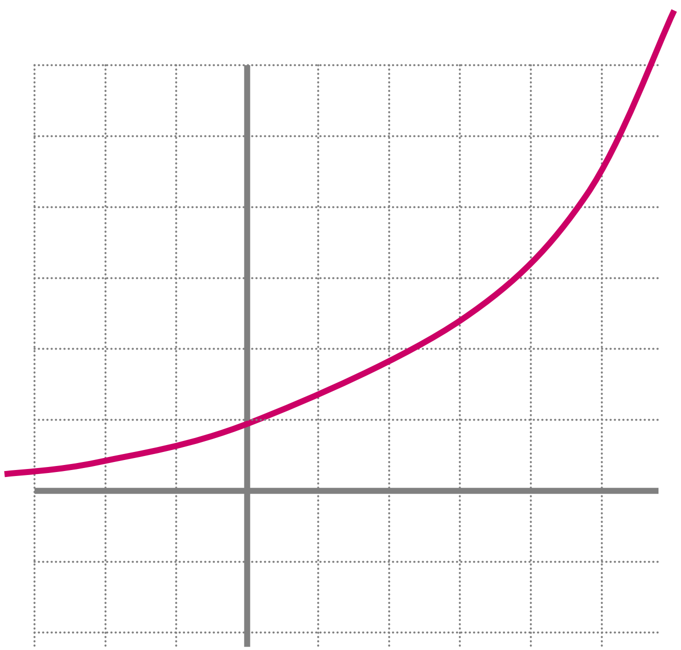


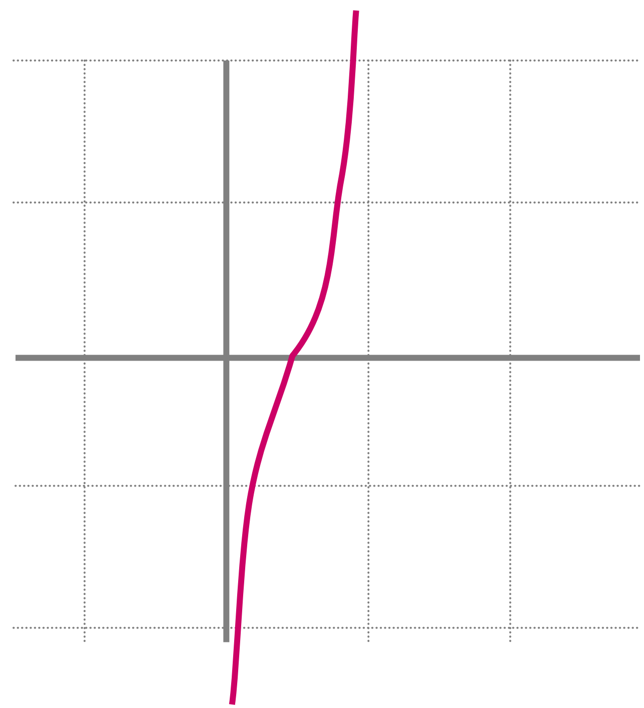
$f: \mathbb{R} \to \mathbb{R}$ ✗

$f: \mathbb{R} \to \mathbb{R}^+$ √

voorbeeld

bijectie tussen $\mathbb{R}$ en $\mathbb{R}^+$
bijectie tussen $\langle 0,1 \rangle$ en $\mathbb{R}$



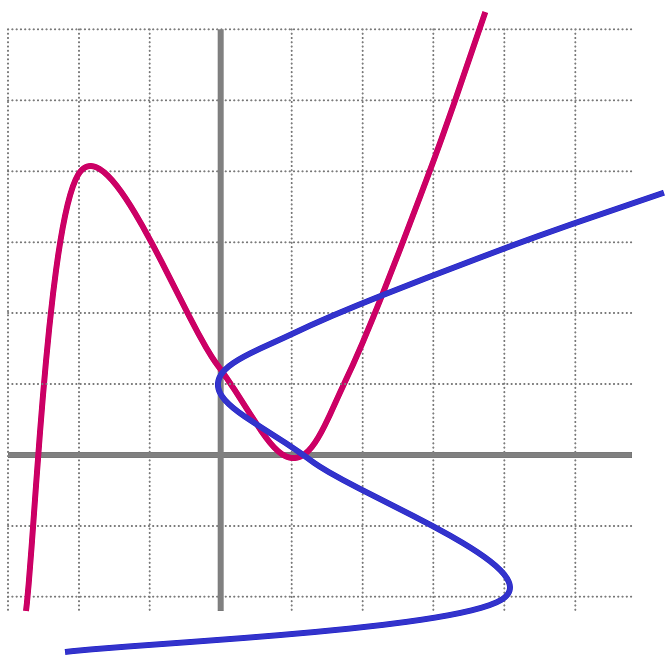


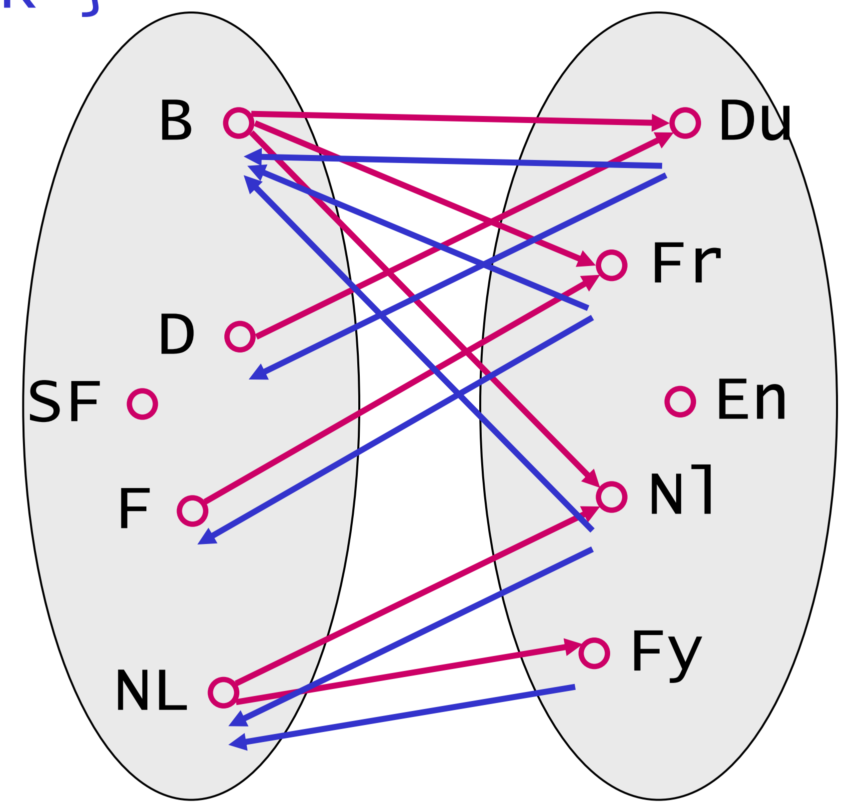
$R \subseteq A \times B$

$R^{-1} \subseteq B \times A$ *inverse relatie*

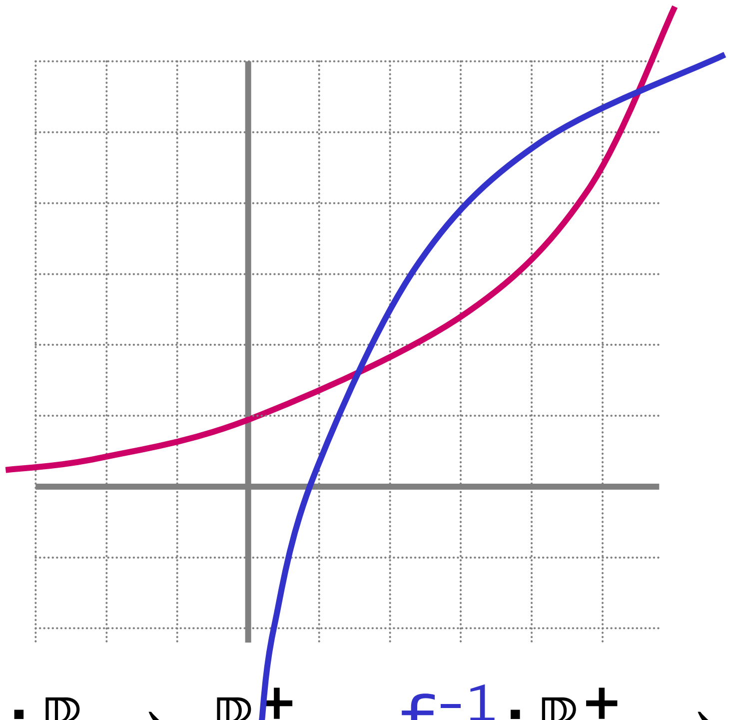
$bR^{-1}a$ desdals aRb.

$R^{-1} = \{ (y,x) \mid (x,y) \in R \}$

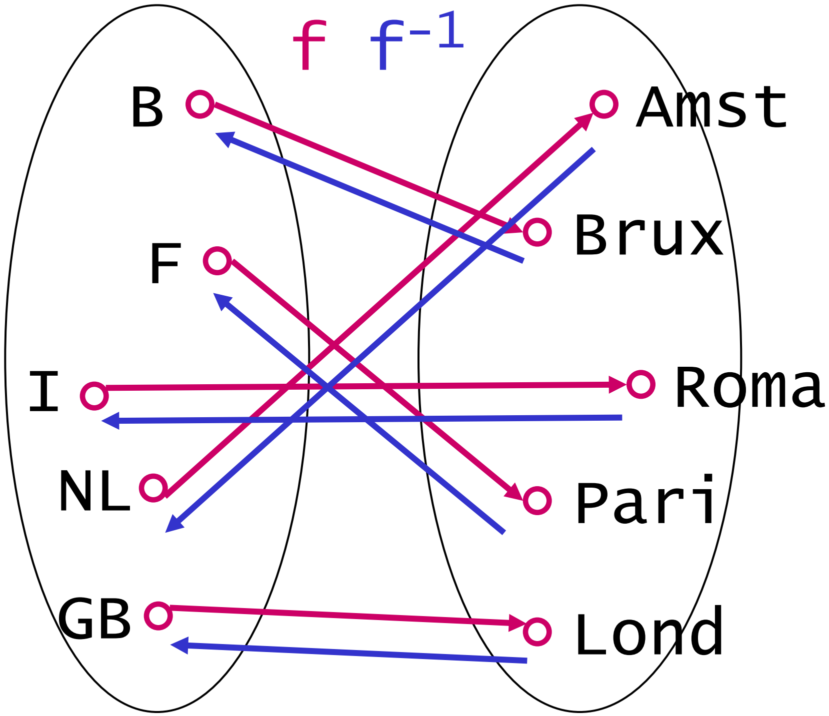




f functie $\Rightarrow$ f^{-1} functie !?



$f: \mathbb{R} \to \mathbb{R}^+$ $f^{-1}: \mathbb{R}^+ \to \mathbb{R}$



dwz. als functie

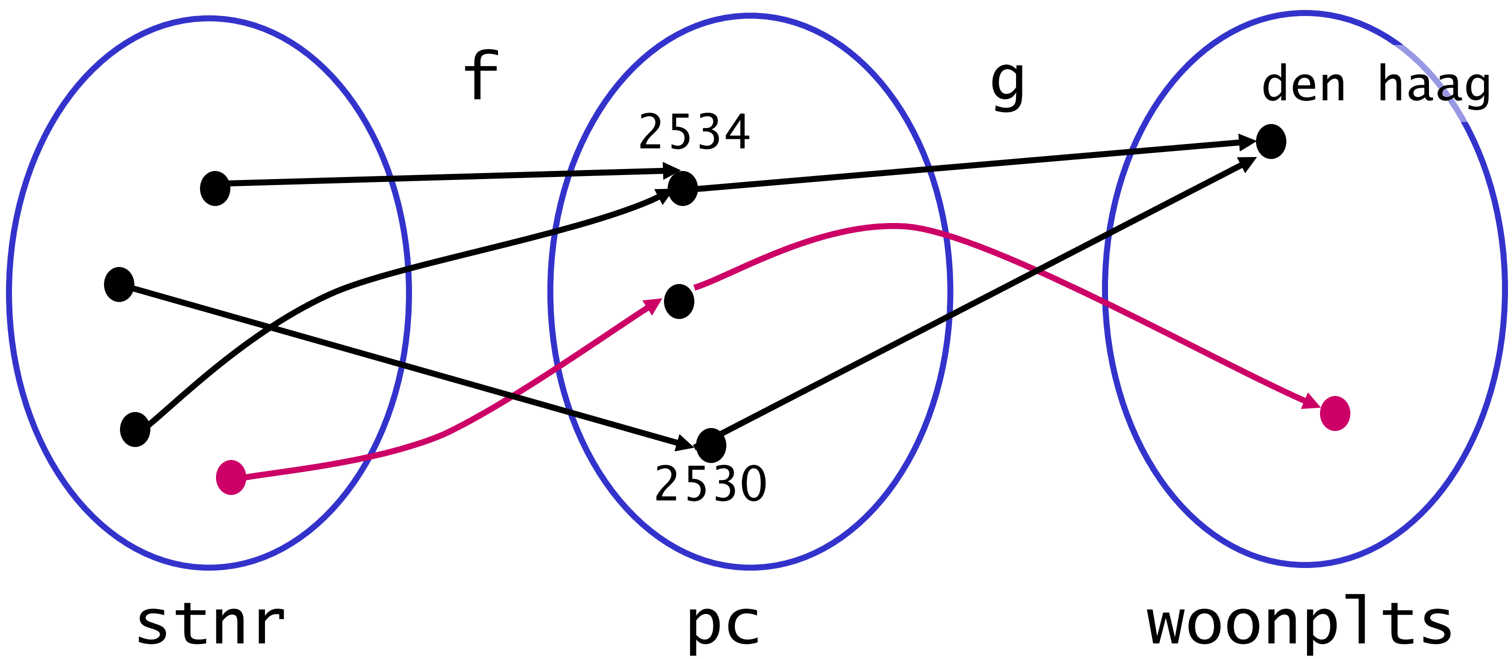
Theorem 3.1

De inverse functie van $f: A \to B$ bestaat desdals f een bijectie is (1-1 en op)

samenstelling

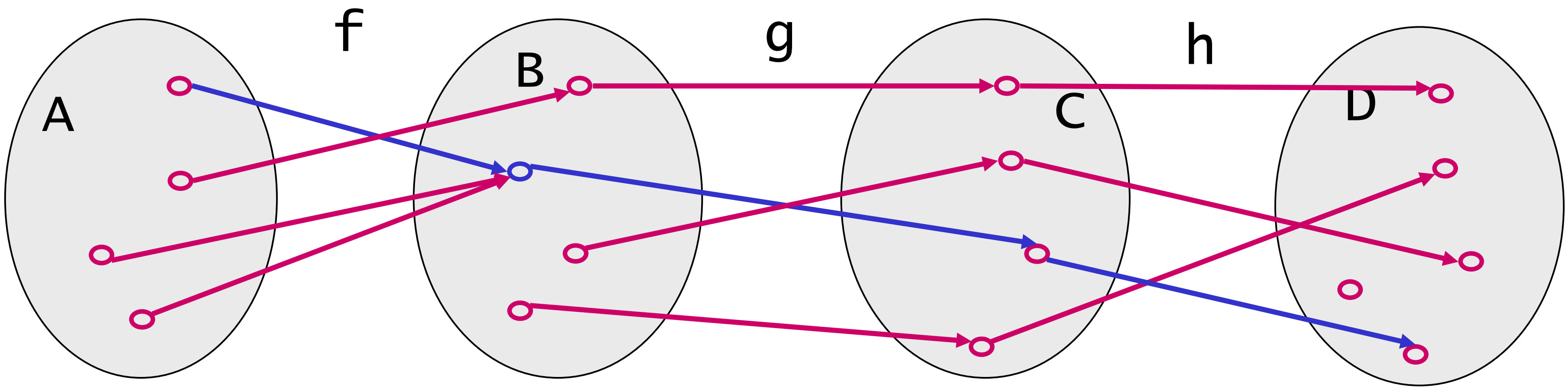
f: A → B en g: B → C functies.
De *samenstelling* van f en g,
genoteerd g∘f (‘g na f’) van A naar C is
gedefinieerd door $(g \circ f)(x) = g(f(x))$ voor alle $x \in A$

volgorde !



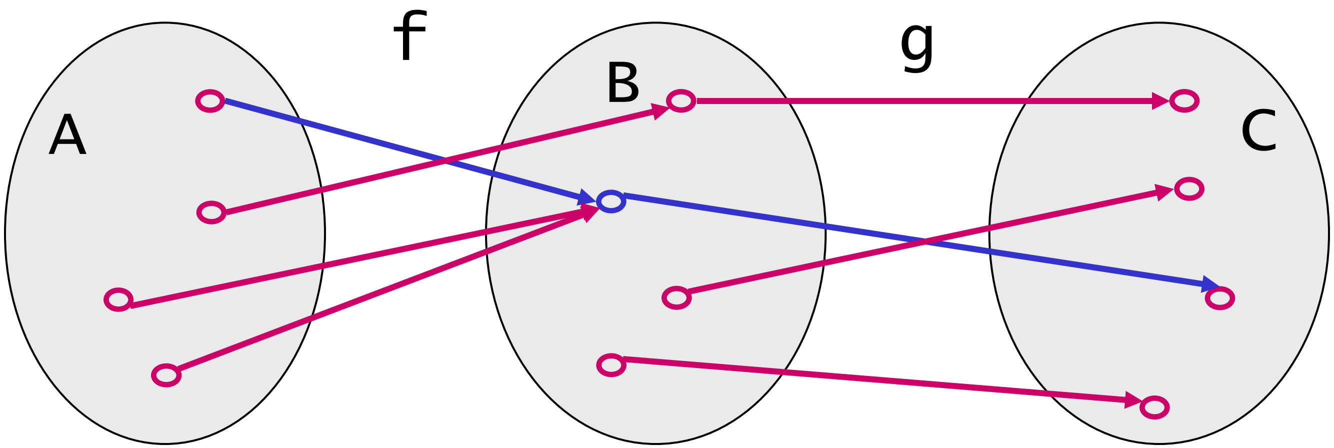
haakjes niet nodig

Samenstellen van functies is associatief:
als f: A → B, g: B → C en h: C → D functies zijn,
dan $(h \circ g) \circ f = h \circ (g \circ f)$.



x R∘S y   y = g∘f (x)

laat f: A → B, g: B → C functies zijn
als f en g surjectief 'op' zijn, dan ook g∘f.
als f en g injectief '1-1' zijn, dan ook g∘f.



Problem 3.9

§3.5 rijen

0,1,0,2,0,1,0,3,0,1,0,2,0,1,0,4, …

$a : \mathbb{N} \to A$ $a : \mathbb{N}^+ \to A$

$a(0),\ a(1),\ a(2),\ \ldots$

$a_0,\ a_1,\ a_2,\ \ldots$ indices

$\{\ a_n \mid n \in \mathbb{N}\ \}$

$(\ a_n\ )_{n \in \mathbb{N}}$

eindige rij (n-tupel)

$a_0,\ a_1,\ a_2,\ \ldots,\ a_n$

$$\sum_{1\le i\le n} a_i = a_1 + a_2 + \cdots + a_n$$

$$\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n-1} a_i + a_n$$

$$\sum_{i=1}^{1} a_i = a_1 , \quad \sum_{i=1}^{0} a_i = 0$$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$\bigcup_{1\le i\le n} A_i$$

$$\bigcup_{i\in I} A_i = \{ x \mid x \in A_i \text{ voor } i \in I \}$$

$a_0 = a$
$a_n = a_{n-1} + v \quad (n \geq 1)$

rij, termen, verschil v

$a_n = a + n \cdot v$

1, 3, 5, 7, 9, 11, 13, …

$$\sum_{i=0}^{n} i = \tfrac{1}{2} n(n+1)$$

reeks:

$$\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} (a + iv) = \tfrac{1}{2}(2a + nv)(n+1)$$

$$\begin{array}{rcccccccccc} S_n & = & 1 & + & 2 & + & \dots & n-1 & + & n & \\ S_n & = & n & + & n-1 & + & \dots & 2 & + & 1 & \\ \hline 2S_n & = & n+1 & + & n+1 & + & \dots & n+1 & + & n+1 & = n(n+1) \end{array}$$

rij: opeenvolgende termen
rekenkundig: verschil termen constant
1,2,3,4,5, …
meetkundig: verhouding constant
1,2,4,8,16, …
reeks: opeenvolgende sommen van rij-elementen

1,2,3,4,5, … -> 1,3,6,10,15, …
1,2,4,8,16, … -> 1,3,7,15,31, …

formule voor rekenkundige reeks:
‘gemiddelde term maal aantal termen’, waarbij
‘gemiddelde’ natuurlijk (eerste + laatste) / 2

$a_0 = a$

$a_n = r \cdot a_{n-1} \qquad (n \geq 1)$

rij, termen, reden $r \in \mathbb{R}$

$a_n = a \cdot r^n$

1, 0.5, 0.25, 0.125, …

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

reeks:

$$\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} a r^i = \frac{a(r^{n+1} - 1)}{r - 1}$$

$$\begin{aligned} rS_n &= \phantom{r^0 +{}} r^1 + r^2 + \ldots r^n + r^{n+1} \\ S_n &= r^0 + r^1 + \ldots r^{n-1} + r^n \\ \hline (r-1)S_n &= r^{n+1} - r^0 \end{aligned}$$

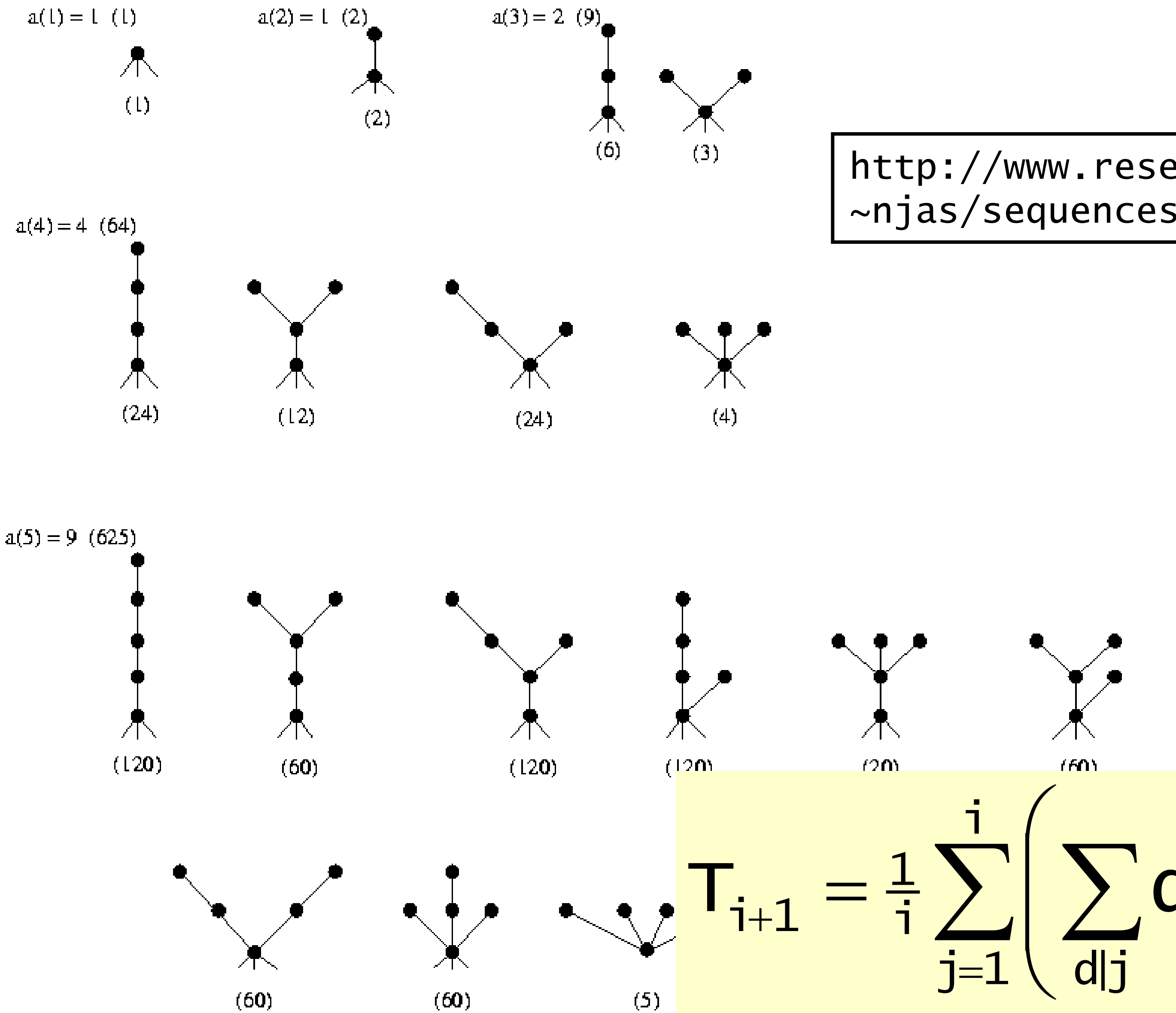
1, 1, 2, 4,

Handbook of Integer Sequences

115, 286,
719, 1842,
4766, 12486,

http://www.research.att.com/~njas/sequences/ **Sloane**

235381,
634847,
1721159,
4688676,
12826228,
35221832,
97055181,
268282855,
743724984,
2067174645,
...
354426847597,



$$T_{i+1} = \frac{1}{i}\sum_{j=1}^{i}\left(\sum_{d|j} d T_d\right) T_{i-j+1}$$

functie $(\forall x \in A)[$
$(\exists y \in B)(\ xfy\ ) \wedge$
$\neg(\exists y \in B)(\exists z \in B)(y \neq z \wedge xfy \wedge xfz)$
$]$

$\forall$ 'voor alle …'
$\exists$ 'er is …'

op $(\forall y \in B)(\exists x \in A)(\ y=f(x)\ )$
$(\forall y)_B(\exists x)_A(\ xfy\ )$

1-1 $(\forall x \in A)(\forall y \in A)(\ f(x)=f(y) \Rightarrow x=y\ )$
$(\forall x)_A(\forall y)_A(\ f(x)=f(y) \Rightarrow x=y\ )$

$\neg(\exists x)_A(\exists y)_A(\ x \neq y \wedge f(x)=f(y)\ )$

? $(\exists y)_B(\forall x)_A(\ y=f(x)\ )$

? $(\exists y)_B(\forall x)_A(\ y \neq f(x)\ )$

elders

⊗

§3.6 recursief gedefinieerde functies

recursively defined:
definition refers to itself

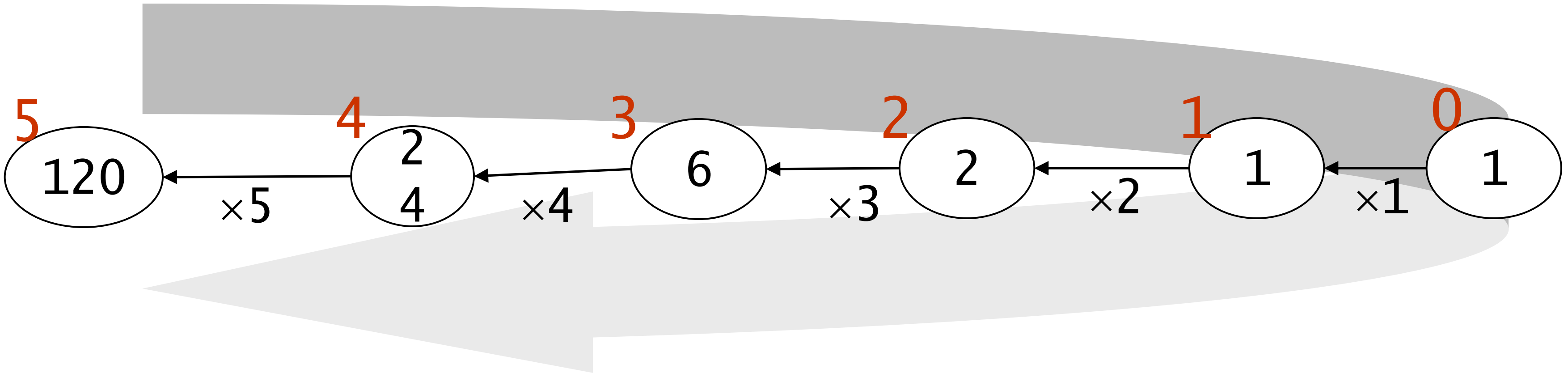
1. base values
2. argument closer to a base value

faculteiten

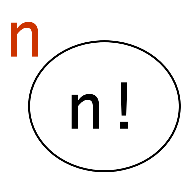
$$0! = 1$$
$$(n+1)! = (n+1) \cdot n!$$

'factorial'

1,1,2,6,24,120,720,5040,40320,362880,3628800,39916800,…



recursief



Sloane A000142

faculteiten

$$0! = 1$$

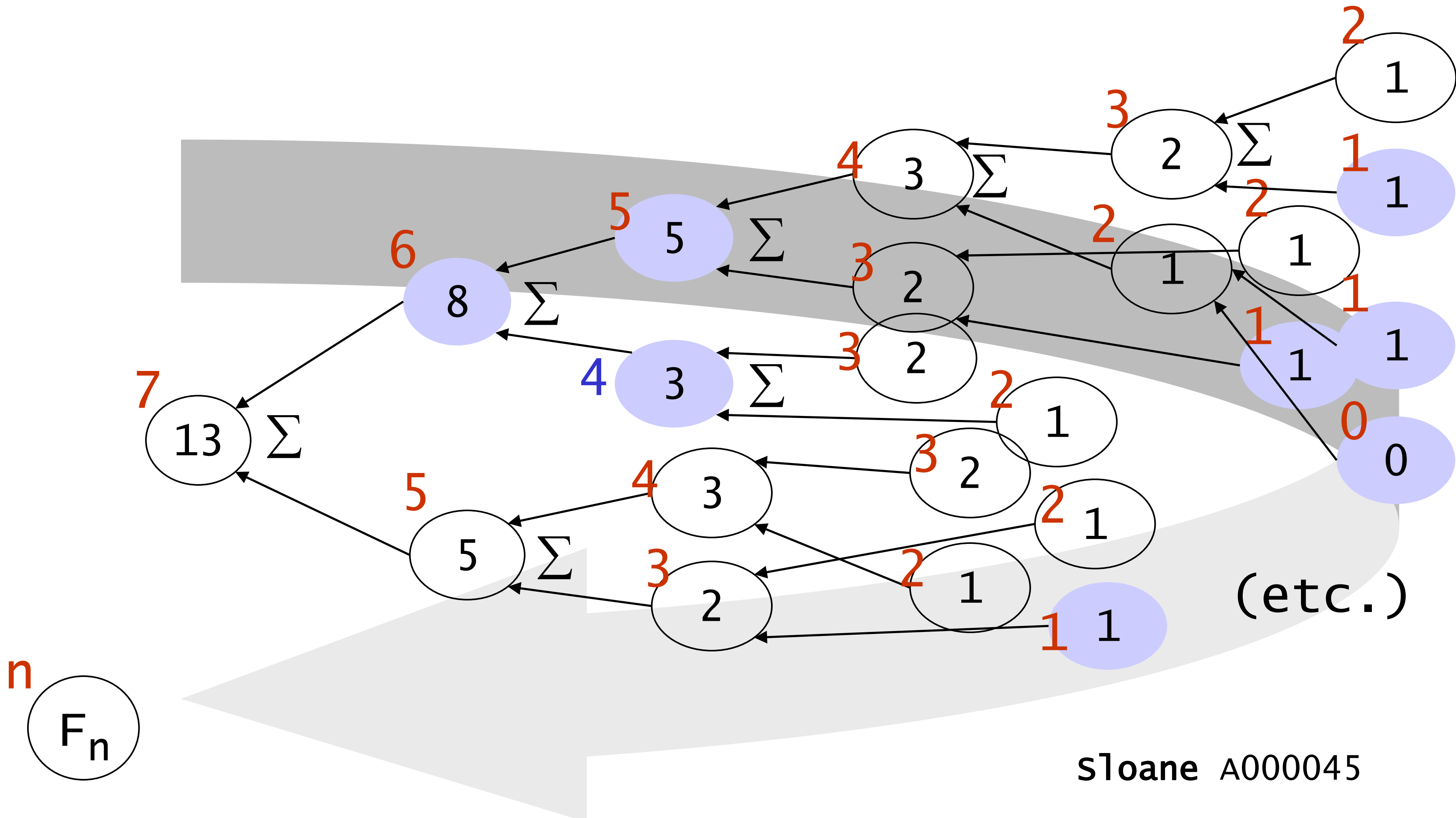
$$(n+1)! = (n+1)\cdot n!$$

```
5! = 5.4!
          4! = 4.3!
                    3! = 3.2!
                              2! = 2.1!
                                        1! = 1.0!
                                                  0! = 1
                                        1! = 1.1 = 1
                              2! = 2.1 = 2
                    3! = 3.2 = 6
          4! = 4.6 = 24
5! = 5.24 = 120
```

Fibonacci

$F_0 = 0 \qquad F_1 = 1$

$F_{n+1} = F_n + F_{n-1} \quad (n \geq 1)$



recursieve definitie

$$f(x) = \begin{cases} x - 10 & \text{als } x > 100 \\ f(\,f(x+11)\,) & \text{als } x \leq 100 \end{cases}$$

in termen van zichzelf
niet inductief (?) …

‘dichter bij basis geval?’

McCarthy 91 function A103847

f(76) =
f(f(87)) =
f(f(f(98))) =
f(f(f(f(109)))) =
f(f(f(99))) =
f(f(f(f(110)))) =
f(f(f(100))) =
f(f(f(f(111)))) =
f(f(f(101))) =
f(f(91)) =
f(f(f(102))) =
f(f(92)) =
… …
f(f(101)) =
f(91) =
f(f(102)) =
f(92) =
… …
f(101) = 91

elders

oneindige verzamelingen ?
'meer/evenveel elementen' $\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}$
$\otimes$

end...