Continuous Probability Spaces

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Continuous Probability Spaces

- We first consider (non-discrete) sample spaces $\Omega \subseteq \mathbb{R}$
- Goal, again, is to be able to compute the probability of events $E \subseteq \Omega$
- This time we don't specify the probability for individual outcomes $m(\omega)$, but a probability density function $f(\omega)$,or usually: $f(x)$
- Definition: Let X be a continuous real-valued random variable. A **density function** for X is a real-valued function that satisfies

$$P(a <= X <= b) = \int_a^b f(x)\,dx$$

for all $a, b \in \mathbb{R}$

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- Read $P(a <= X <= b)$ as $P(E)$, with $E = \{\omega | \omega \in \Omega, a \leq \omega \leq b\}$
- A density function is a density in the sense that it gives the probability **per unit sample space**
- Analogy: mass density of a wire:

  Suppose we have a wire and its mass **density** along its length is given by f(x)
- Example 1: we have a wire of 2 meters long with a uniform density of 10 kg/m2. Draw graph. Explain some masses.
- Example 2: Now for a general density function f(x). Now

$$M \approx \sum_{i=1}^{N} f(x_i)\Delta x$$

- We can get the exact mass by letting $\Delta x \to 0$:

$$M = \int_a^b f(x)\, dx$$

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- Using the density function we can compute the probability of (almost) any (reasonable) event $E \subseteq \Omega$:

$$P(E) = \int_E f(x)\, dx$$

- Note:

$$P(\Omega) = \int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\mathrm{P}(\{x\}) = \int_x^x f(s)\, ds = 0$$

Two examples of density functions

- Uniform distribution on an interval [a, b]:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if} \quad a \leq x \leq b \\ 0 & \quad\quad \text{elsewhere} \end{cases}$$

- Exponential distribution: often a good model for times between occurrences

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{if} \quad t \geq 0 \\ 0 & \text{if} \quad t < 0 \end{cases}$$

Uniform density for 2 variables

- Now consider $\Omega \subseteq \mathbb{R}^2$
- For a uniform density $f(x,y) = \frac{1}{\text{Area}(\Omega)}$

- Probability of an event $E \subseteq \Omega$:

$$P(E) = \int\int_E f(x)\,dx\,dy = \frac{\text{Area}(E)}{\text{Area}(\Omega)}$$

- Dart example: compute the probability that the dart lands in a certain region (e.g. first quadrant, half slice near rim 3/16)

Cumulative distribution functions

- Let X be a continuous real-valued random variable with density function f(x). The cumulative distribution function F(x) is defined by $$F(x) = P(X <= x) = \int_{-\infty}^{x} f(t)\, dt$$
- F(x) is a specific primitive of f(x): $\frac{d}{dx}F(x) = f(x)$
- Uses of the cumulative distribution function:
  - Sometimes easier to determine than the density function
  - It's already integrated out making it easier to use to compute probabilities:

$$P(X <= a) = F(a)$$

$$P(X > a) = 1 - F(a)$$

$$P(a \leq X \leq b) = F(b) - F(a)$$

e.g.

$$P(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx = \int_{-\infty}^{b} f(x)\, dx - \int_{-\infty}^{a} f(x)\, dx$$

Cumulative distribution functions

- The uniform density

$$F(x) = P(X <= x) = \int_{-\infty}^{x} \frac{1}{b-a}\, dt = \frac{x-a}{b-a}$$

- The exponential density

$$F(x) = P(X <= x) = \int_{-\infty}^{x} \lambda e^{-\lambda t}\, dt = \left[-e^{-\lambda t}\right]_{-\infty}^{x} = 1 - e^{-\lambda t}$$

Assignment

The density of a continuous random variable X is given by

$$f(x) = \begin{cases} x & \text{if} \quad 0 < x < 1 \\ \frac{1}{2} & \text{if} \quad 1 < x < 2 \\ 0 & \quad \text{elsewhere} \end{cases}$$

(a) Compute the cumulative distribution function F(x)

(b) Compute P({ x > 3/2 })

(c) Compute P({1/2 < x < 3/2})