Theorem: $\forall n \in \mathbb{N}: \quad 1+2+\dots+n = \frac{1}{2} n(n+1)$

Proof:

Let $P(n): \quad 1+2+\dots+n = \frac{1}{2} n(n+1)$
for $n \in \mathbb{N}$.
$P(1): \quad 1 = \frac{1}{2} \cdot 1(1+1) = 1$ is true.
Assume for $n \in \mathbb{N}$ given, $P(n)$ is true.
Then $1+2+\dots+n+(n+1) = (1+2+\dots+n)+(n+1)$

$$= \frac{1}{2} n(n+1) + (n+1) \qquad \text{(as } P(n) \text{ is true)}$$

$$= (\tfrac{1}{2} n+1)(n+1) = \tfrac{1}{2}(n+2)(n+1)$$

$$= \tfrac{1}{2}(n+1)(n+2) = \tfrac{1}{2}(n+1)((n+1)+1).$$

Hence $P(n+1)$ is true.
Thus we proved that $\forall n \in \mathbb{N}: P(n) \Rightarrow P(n+1)$.
Hence by induction we proved our theorem. $\boxtimes$

$$1^2+2^2+\dots+n^2 \overset{?}{=} an^3+bn^2+cn+d$$

For $n = 1,2,3,4$ we have by solving the equations:
$a = \frac{1}{3}$, $b = \frac{1}{2}$, $c = \frac{1}{6}$, $d = 0$

Theorem: $\forall n \in \mathbb{N}: \quad 1^2+2^2+\dots+n^2 = \frac{1}{3}n^3+\frac{1}{2}n^2+\frac{1}{6}n$

$$= \tfrac{1}{6} n(n+1)(2n+1)$$

Proof: $P(1)$ is true. From $P(n)$ true it follows that $1^2+2^2+\dots+n^2+(n+1)^2 = \frac{1}{6} n(n+1)(2n+1)+(n+1)^2$

$= \frac{1}{6}(n+1)(n+2)(2n+3) = \frac{1}{6}(n+1)((n+1)+1)(2(n+1)+1)$,
i.e., $P(n+1)$ is true.
Hence we proved our theorem by induction. $\boxtimes$

<u>Theorem:</u> $\forall n \in \mathbb{N}, \; \forall x \in \mathbb{R}$ with $x \neq 1$:

$$\sum_{k=1}^{n-1} x^k = 1 + x + \ldots + x^{n-1} = \frac{1-x^n}{1-x}$$

<u>Proof:</u> $P(1)$ true. Assume $P(n)$ is true.

Then

$$\sum_{k=1}^{n} x^k = 1 + x + \ldots + x^{n-1} + x^n$$

$$= \frac{1-x^n}{1-x} + x^n \quad (P(n) \text{ is true})$$

$$= \frac{1-x^n}{1-x} + \frac{(1-x)x^n}{(1-x)}$$

$$= \frac{1-x^n+x^n-x^{n+1}}{1-x}$$

$$= \frac{1-x^{n+1}}{1-x}.$$

It follows that $P(n+1)$ is true.

Hence the theorem is proved by induction. $\boxtimes$

<u>Theorem:</u> $\forall a \in \mathbb{R}, \; a \geq -1, \; \forall n \in \mathbb{N}: (1+a)^n \geq 1 + n.a$.

<u>Proof:</u>

Homework!

Differentiate Product Rule: $(f(x)\cdot g(x))' = f'(x)g(x) + f(x)\cdot g'(x)$

<u>Theorem:</u> $\forall n \in \mathbb{N}$: if $f(x) = x^n$, then $f'(x) = n\,x^{n-1}$

<u>Proof:</u>

Let $P(n)$: if $f(x) = x^n$, then $f'(x) = n\,x^{n-1}$

Then $P(1)$: if $f(x) = x$, then $f'(x) = \frac{dx}{dx} = 1 = 1\cdot x^{1-1} = 1$.

For $g(x) = x^{n+1}$ we have $g'(x) = (x^{n+1})' =$

$$(x^n\cdot x)' = (x^n)'\cdot x + x^n\cdot (x)'$$
$$= n\,x^{n-1}\cdot x + x^n\cdot 1$$
$$= n\,x^n + x^n$$
$$= (n+1)\,x^n.$$

$\Rightarrow$ $P(n+1)$ is true. $\Rightarrow$

Theorem proved by induction. $\boxtimes$

<u>Notations</u>

(n faculty) $\quad n! = 1\cdot 2\cdot \ldots \cdot n \quad ; \quad 0! \underset{\text{def.}}{=} 1$

(n over k) $\quad \binom{n}{k} = \dfrac{n(n-1)\cdot \ldots \cdot (n-k+1)}{k!}$

$$= \frac{n!}{k!\,(n-k)!} \quad ; \quad \binom{n}{0} \underset{\text{def.}}{=} 1$$

Using this notation we can prove the following theorem.

<u>Theorem:</u> Let $f(x) = x^n$, then for $1 \le k \le n$, the k-th derivative of $f(x)$: $f^{(k)}(x) = k!\binom{n}{k}x^{n-k}$.

<u>Proof:</u>

By induction ....

Theorem: (Binomium of Newton)

Let $a, b \in \mathbb{R} \setminus \{0\}$. $\forall n \in \mathbb{N}$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = a^n + \binom{n}{1} a^{n-1} b + \dots + \binom{n}{n-1} a b^{n-1} + b^n$$

Proof: By induction.

Hint: use the property $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ which explains the numbers in the triangle of Pascal depicting the binomial coefficients for $(a+b)^n$:

Pascal's Triangle

$n=0$:							1						
$n=1$:						1		1					
$n=2$:					1		2		1				
$n=3$:				1		3		3		1			
$n=4$:			1		4		6		4		1		
$n=5$:		$k=$ 1_0		5_1		$\boxed{10_2}$		$\boxed{10_3}$		5_4		1_5	
$n=6$:	1		6		15		$\boxed{20_3}$		15		6		1
	$k=0$		1		2				4		5		6

$$\binom{5+1}{3} = \binom{5}{3} + \binom{5}{2} = \frac{120}{12} + \frac{120}{12} = 10 + 10 = 20$$

Also note: $\binom{n}{k} = \binom{n}{n-k}$.

.... ☒

Now $(1+a)^n = 1 + na + \frac{1}{2}n(n-1)a^2 + \dots + a^n$.

If $a \geq 0$, then it is clear that:

$(1+a)^n \geq na \quad (a \geq 0)$

$(1+a)^n \geq 1 + na \quad (a \geq 0)$

$(1+a)^n \geq \frac{1}{2}n(n-1)a^2 \quad (a \geq 0)$ etc.

<u>Definition:</u> Let (a_n): $a_1, a_2, a_3, \dots$ an infinite row of numbers in $\mathbb{R}$. Let $\varepsilon \in \mathbb{R}$.
(a_n) has <u>limit</u> ℓ if $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ such that $\forall n > N$: $|a_n - \ell| < \varepsilon$.

<u>Notation:</u> $\lim\limits_{n \to \infty} a_n = \ell$; $\lim a_n = \ell$; ...

<u>Th.</u> Let $r \in \mathbb{R}$, with $|r| < 1$ given. Then $\lim\limits_{n \to \infty} r^n = 0$.

<u>Proof:</u>
Let $r \in \mathbb{R}$, with $|r| < 1$. We can write $|r| = (1+a)^{-1} = \frac{1}{1+a}$ with $a > 0$.
Then $|r^n - 0| = |r^n| = \left(\frac{1}{1+a}\right)^n = \frac{1}{(1+a)^n}$

$$\leq \frac{1}{na} < \varepsilon \quad \text{if } n > \frac{1}{a\varepsilon}. \; (*)$$

$(1+a)^n \geq na$ ☒

(*) Let $\varepsilon > 0 \Rightarrow$ take $N = \left\lceil \frac{1}{a\varepsilon} \right\rceil$.
Then $\forall n > N$: $|r^n - 0| < \varepsilon$.

<u>Theorem:</u> a) $\lim\limits_{n \to \infty} \sqrt[n]{a} = 1$ if $a \in \mathbb{R}$, $a > 0$

b) $\lim\limits_{n \to \infty} \sqrt[n]{n} = 1$

‖ Note:- If (a_n) has limit ℓ, then $a_N, a_{N+1}, a_{N+2}, \ldots$ also has limit ℓ.

- If $a_n \to \ell$ and $b_n \to m$,
  then
  - $a_n + b_n \to \ell + m$
  - $a_n \cdot b_n \to \ell \cdot m$
  - $a_n \cdot b_n^{-1} \to \ell \cdot m^{-1}$, if $m \neq 0$

- If $a_n \geq 0$ and $a_n \to \ell$,
  then $\sqrt{a_n} \to \sqrt{\ell}$

‖ The n^{th} derivative of the product of two functions.

We can prove: $(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} \cdot g^{(k)}$,

where $f^{(0)} = f$, and $g^{(0)} = g$

Now, if $f(x) = x^2 \sin x$, then

$$f^{(n)}(x) = x^2 \sin x - 2nx \cos x - n(n-1) \sin x$$
if $n = 4k \quad (k \in \mathbb{N})$

Question: what are the formulas for $n = 4k+1$, $n = 4k+2$, $n = 4k+3$?

Note: $n!$ is the number of ways that n objects can be placed in an ordered row, i.e., the number of permutations of n objects.

Proof by induction:

$P(1)$ true.

Assume $P(n)$ is true, then for the $(n+1)^{th}$ object we can places this on $(n+1)$-ways within any of the ordered n objects. $\Rightarrow (n+1)\,n!$ orderings.
$= (n+1)!$ orderings.

$\binom{n}{k}$ is equal to the number of ways we can choose k objects out of n objects without taking attention of the ordering of the selection:

Theorem III Assume $P(n)$ a logical expression in which an $n \in \mathbb{N}$ appears. Assume $P(1)$ is true, and that $\forall\, n \in \mathbb{N}$ If $P(1), \ldots, P(n)$ true $\Rightarrow P(n+1)$ true. Then $P(n)$ is true for every $n \in \mathbb{N}$.

Definition: Fibonacci numbers.

$F_0 = F_1 = 1$

$F_n = F_{n-2} + F_{n-1} \qquad n \in \mathbb{N},\ n \geq 2$

Theorem: Let α and β the roots of $x^2 - x - 1 = 0$.

$\Rightarrow \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$

Then $F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$

Proof: By induction (IIIth form).

$$F_{n+1} = F_n + F_{n-1} = \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}$$

$$= \frac{\alpha^n - \beta^n + \alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}$$

$$= \frac{\alpha^n + \alpha^{n-1} - (\beta^n + \beta^{n-1})}{\alpha - \beta}$$

$$= \frac{\alpha^{n+1}\left(\frac{1}{\alpha} + \frac{1}{\alpha^2}\right) - \beta^{n+1}\left(\frac{1}{\beta} + \frac{1}{\beta^2}\right)}{\alpha - \beta}$$

Note $\frac{1}{\alpha} + \frac{1}{\alpha^2} = \frac{1+\alpha}{\alpha^2}$. Furthermore α is root of the equation $x^2 - x - 1 = 0 \Rightarrow \alpha^2 = 1 + \alpha$.

$$\Rightarrow \frac{1+\alpha}{\alpha^2} = 1$$

This holds also for β.

$$\Rightarrow F_{n+1} = \frac{\alpha^{n+1}\left(\frac{1}{\alpha} + \frac{1}{\alpha^2}\right) - \beta^{n+1}\left(\frac{1}{\beta} + \frac{1}{\beta^2}\right)}{\alpha - \beta} = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}.$$

$\Rightarrow P(n+1)$ is also true.

Theorem proved by induction (III) $\boxtimes$