Solutions of the exercises of Chapter 9 for

Theorie van Concurrency

9.1 a) Transition t_2 cannot fire because it has no input concession, and transition t_1 cannot fire because it has no output concession (p_3 has capacity 2). Transition t_3 can fire, from configuration $(1, 0, 2, 0)$ to configuration $(1, 1, 0, 2)$.

b) See Fig.1. Places q_3 and q_4 have been added to M.

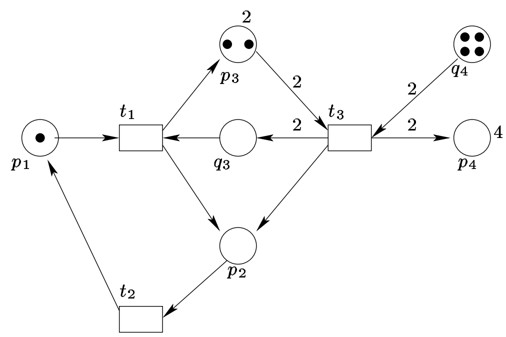

Figure 1: Answer to ex. 9.1(b)

c) See Fig.1, from which the capacities of places p_3 and p_4 are removed (i.e., made ω).

9.2 We have to show that for no p there exists $k \in \mathbb{N}$ so that $C(p) \leq k$ for all $C \in \mathbb{C}_M$. Considering Lemma 161 we have to show that the configurations of M increase. So we have to consider the SCG:

$$\rightarrow (1, 0, 0, 0) \xrightarrow{t_1} (0, 1, 1, 0) \xrightarrow{t_2} (0, 0, 2, 0) \xrightarrow{t_3} (2, 0, 0, 1) > (1, 0, 0, 0)$$

Hence, by Lemma 161, $C_{in} = (1, 0, 0, 0)\ [(t_1 t_2 t_3)^n\rangle\ (n+1, 0, 0, n) \in \mathbb{C}_M$ for every n. This shows that p_1 and p_4 are not bounded. Now $(n+1, 0, 0, n)\ [t_1^n\rangle\ (1, n, n, n)$, which shows that p_2 and p_3 are not bounded.

9.3 a) To apply the algorithm it may be convenient to use Lemma 191 and the matrix of M, instead of looking at the figure of M. The matrix is $\underline{M} = \begin{pmatrix} -1 & +1 & 0 & 0 \\ -1 & -1 & +1 & 0 \\ -1 & 0 & 0 & +1 \\ +3 & 0 & -1 & -1 \end{pmatrix}$, where the rows correspond to t_1, t_2, t_3, t_4 from top to bottom, and the columns to p_1, p_2, p_3, p_4 from left to right.

$\mathbb{C}_M$ is finite: see Fig.2, except the bold part.

b) See the bold part in Fig.2. Since $(3, 1, 0, 0) > (2, 1, 0, 0)$, $\mathbb{C}_{M'}$ is infinite (i.e., M' is unbounded) according to Lemma 162.

9.4 The matrix of M is $\underline{M} = \begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & -2 & 1 \end{pmatrix}$.

A vector $i = (x_1, x_2, x_3, x_4)$ is a p-invariant of M if $\underline{M} \cdot i = 0$, i.e., if

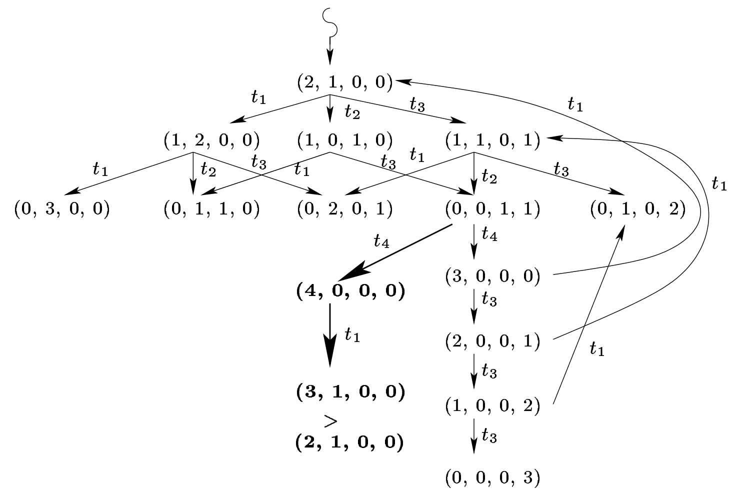

Figure 2: Answer to ex. 9.3

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The corresponding system of linear equations is: $\begin{cases} -x_1 + x_2 + x_3 = 0 \\ -x_2 + x_3 = 0 \\ 2x_1 - 2x_3 + x_4 = 0 \end{cases}$,

which can be simplified to $\begin{cases} x_1 = x_2 + x_3 = 2x_3 \\ x_2 = x_3 \\ x_4 = 2x_3 - 2x_1 = -2x_3 \end{cases}$.

So, writing $x_3 = \lambda$, the p-invariants are $\begin{pmatrix} 2\lambda \\ \lambda \\ \lambda \\ -2\lambda \end{pmatrix} = \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \\ -2 \end{pmatrix}$, $\lambda \in \mathbb{Z}$.

Thus, there is essentially just one p-invariant: $(2, 1, 1, -2)$.

9.5 a) As observed in the answer to Exercise 9.3, $\underline{M} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 3 & 0 & -1 & -1 \end{pmatrix}$. Solving the system we get $\lambda(1, 1, 2, 1)$, $\lambda \in \mathbb{Z}$.

b) $\underline{M} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 4 & 0 & -1 & -1 \end{pmatrix}$ and solving the system we only get the trivial p-invariant $(0, 0, 0, 0)$.

9.6 a) $\underline{M} = \begin{pmatrix} -5 & 1 & 5 & 2 & 0 \\ -3 & -1 & -1 & 0 & 2 \\ 4 & 0 & -2 & -1 & -1 \end{pmatrix}$.

Solving the system of equations we see that $\underline{t}_1 + \underline{t}_2 = -2 \cdot \underline{t}_3$ and so one of the rows can be removed. We remove the first row. The remaining system is:

$$\begin{aligned} x_2 &= -3x_1 - x_3 + 2x_5 \\ x_4 &= 4x_1 - 2x_3 - x_5 \end{aligned}$$

Thus, writing $x_1 = \lambda$, $x_3 = \mu$, and $x_5 = \nu$, the solution is

$$(\lambda, -3\lambda - \mu + 2\nu, \mu, 4\lambda - 2\mu - \nu, \nu)$$

for $\lambda, \mu, \nu \in \mathbb{Z}$.

So, all the p-invariants are: $\lambda \begin{pmatrix} 1 \\ -3 \\ 0 \\ 4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ -1 \\ 1 \\ -2 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$ with $\lambda, \mu, \nu \in \mathbb{Z}$.

b) Just looking: for $\lambda = 2$, $\mu = 1$, and $\nu = 4$ we get the positive invariant
$\begin{pmatrix} 2 \\ -6 \\ 0 \\ 8 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 8 \\ 0 \\ -4 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \\ 4 \end{pmatrix}$ which covers M.
So M is bounded, by Theorem 198. Fig.3 represents another way to prove that M is bounded.

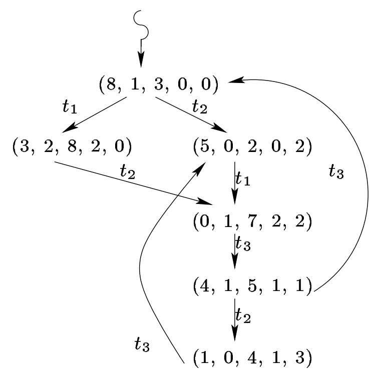

Figure 3: Answer to ex. 9.6(b)

c) The invariant $i = (1, -3, 0, 4, 0)$ has value $C_{in} \cdot i = (8, 1, 3, 0, 0) \cdot (1, -3, 0, 4, 0) = 8 - 3 = 5$. But $C \cdot i = (2, 2, 2, 2, 2) \cdot (1, -3, 0, 4, 0) = 2 - 6 + 8 = 4$. Hence $C \notin \mathbb{C}_M$, by Lemma 189.

9.7 a) $\underline{M} = \begin{pmatrix} -3 & -1 & 1 & 1 \\ -1 & 3 & -1 & 0 \\ 4 & -2 & 0 & -1 \end{pmatrix}$. Writing $x_1 = \lambda$ and $x_2 = \mu$, we obtain the p-invariants
$\lambda \begin{pmatrix} 1 \\ 0 \\ -1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 3 \\ -2 \end{pmatrix}$ with $\lambda, \mu \in \mathbb{Z}$.
Taking $\lambda = \mu = 1$ we get the positive invariant $i = (1, 1, 2, 2)$, which covers M.
If asked to draw the space of the positive p-invariants, we obtain the intersection of the four areas $\lambda > 0$, $\mu > 0$, $-\lambda + 3\mu > 0$ (i.e., $\lambda < 3\mu$, i.e., $\lambda/\mu < 3$), and $4\lambda - 2\mu > 0$ (i.e., $\mu < 2\lambda$, i.e., $\lambda/\mu > 1/2$). This is drawn in Fig.4 where only the black dots are considered as we are in $\mathbb{Z}$ and not in $\mathbb{R}$.

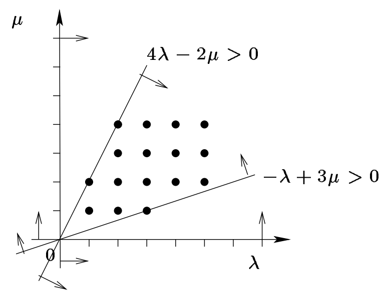

Figure 4: Added answer to ex. 9.7(a)

b) The value of i is $C_{in} \cdot i = (6\ 1\ 2\ 0) \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} = 11$. Hence, by Lemma 189, $C(p_1) + C(p_2) + 2C(p_3) + 2C(p_4) = 11$ for all $C \in \mathbb{C}_M$. Since $C(p_1) + C(p_2) \geq 0$, we obtain that $2C(p_3) + 2C(p_4) \leq 11 < 12$, and so $C(p_3) + C(p_4) < 6$.
The value of p-invariant $(1, 0, -1, 4)$ is 4, and so $C(p_1) - C(p_3) + 4C(p_4) = 4$ for all $C \in \mathbb{C}_M$. Adding to this that $C(p_3) + C(p_4) < 6$, we get that $C(p_1) + 5C(p_4) < 10$ and so $5C(p_4) < 10$, i.e., $C(p_4) < 2$. Hence $C(p_4) \leq 1$.

c) Since M is covered by positive p-invariants, $SCG(M)$ is finite (by Theorem 198). It is drawn in Fig.5.

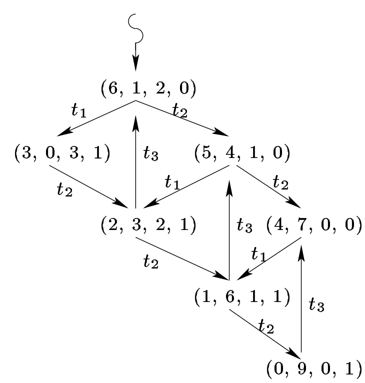

Figure 5: Answer to ex. 9.7(c)

9.8 Proof of Lemma 195.

We first observe that for an arbitrary $i : P \to \mathbb{Z}$, i is a p-invariant iff

$$\sum\{W(p,t) \cdot i(p) \mid p \in {}^\bullet t,\ i(p) \neq 0\} = \sum\{W(t,p) \cdot i(p) \mid p \in t^\bullet,\ i(p) \neq 0\} \text{ for all } t \in T.$$

(by Theorem 192(2) and Definition 190(1)). Note, by the way, that this implies Corollary 193.

Now let i be a positive p-invariant and $S = \{p \in P \mid i(p) \neq 0\}$. By the above observation,

$$\sum\{W(p,t) \cdot i(p) \mid p \in {}^\bullet t \cap S\} = \sum\{W(t,p) \cdot i(p) \mid p \in t^\bullet \cap S\} \text{ for all } t \in T.$$

Since i is positive, all numbers $W(p,t) \cdot i(p)$ and $W(t,p) \cdot i(p)$ are positive. Hence, either both sets of numbers are empty, or both are nonempty. In other words, for all $t \in T$, ${}^\bullet t \cap S \neq \emptyset \Leftrightarrow t^\bullet \cap S \neq \emptyset$. This means that ${}^\bullet S = S^\bullet$ (cf. the remarks following Lemma 47).

9.9 a) $\underline{M} = \begin{pmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$ so we get the p-invariants $\lambda \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ with $\lambda, \mu \in \mathbb{Z}$.

For the characteristic p-invariants we must have $\lambda, \mu \in \{0, 1\}$. Considering these possibilities we obtain $i_1 = (1,0,1,0,1,0)$, $i_2 = (0,1,0,1,0,1)$, $i_3 = (0,0,0,0,0,0)$, and $i_4 = (1,1,1,1,1,1)$.

Since $C_{in} = (1,1,0,0,0,0)$, the values of these characteristic p-invariants are $C_{in} \cdot i_1 = 1$, $C_{in} \cdot i_2 = 1$, $C_{in} \cdot i_3 = 0$, and $C_{in} \cdot i_4 = 2$. Hence i_1 and i_2 are the sequential components of M (see Definition 188); they are the characteristic functions of $\{p_1, p_3, p_5\}$ and $\{p_2, p_4, p_6\}$.

b) This follows from p-invariant i_4 with value 2, by Lemma 189.

9.10 a) $\alpha_1 = (p_1, p_5, p_6, p_2)$, $\alpha_2 = (p_3, p_4)$, $\alpha_3 = (p_1, p_3, p_2)$, and $\alpha_4 = (p_4, p_5, p_6)$.

b) According to Theorem 204, M is live iff all cycles have value > 0. The values of the cycles are: $C_{in}(\alpha_1) = 1$, $C_{in}(\alpha_2) = 2$, $C_{in}(\alpha_3) = 1$, and $C_{in}(\alpha_4) = 2$. So M is live (and hence reduced).

Since M is reduced, Theorem 205 is applicable. Hence M is safe iff every place of M belongs to a cycle of M with value 1. This does not hold for M as p_4 does not belong to α_1 or α_3. So M is not safe. Note that, indeed, $C_{in}[t_3 t_4\rangle C$ with $C(p_4) = 2$.

c) If $C(p_3) = C(p_4) = 1$ then α_2 has value 2. Hence, liveness and safeness of (P, T, F, W, C) implies by Theorem 205 that both α_3 and α_4 have value 1 (because of places p_3 and p_4, respectively). But then α_1 has value 0, contradicting Theorem 204. Hence such a configuration C does not exist.

9.11 By Definition 210, $S \subseteq P_M$ is a siphon if ${}^\bullet S \subseteq S^\bullet$, i.e., $t^\bullet \cap S \neq \emptyset$ implies ${}^\bullet t \cap S \neq \emptyset$ for all $t \in T$. In this case that means the following:
For t_4 and t_5: $p_3 \in S \Leftrightarrow p_5 \in S$. For t_6 and t_7: $p_4 \in S \Leftrightarrow p_6 \in S$.
For t_2: $p_1 \in S \Rightarrow p_3 \in S$. For t_3: $p_2 \in S \Rightarrow p_4 \in S$.
For t_1: $[p_3 \in S \Rightarrow (p_1 \in S \text{ or } p_2 \in S)]$ and $[p_4 \in S \Rightarrow (p_1 \in S \text{ or } p_2 \in S)]$.

Thus the siphons are: $\emptyset, P_M, \{p_1, p_3, p_5\}, \{p_2, p_4, p_6\}, \{p_1, p_3, p_4, p_5, p_6\}, \{p_2, p_3, p_4, p_5, p_6\}$.

Similarly, $S \subseteq P_M$ is a trap if $S^\bullet \subseteq {}^\bullet S$, i.e., ${}^\bullet t \cap S \neq \emptyset$ implies $t^\bullet \cap S \neq \emptyset$ for all $t \in T$. Due to the symmetries in the net, the conditions for t_1, t_2, t_3 are the same as for siphons except that p_1 and p_3 are interchanged and so are p_2 and p_4.

Hence the traps are: $\emptyset, P_M, \{p_1, p_3, p_5\}, \{p_2, p_4, p_6\}, \{p_1, p_2, p_3, p_5\}, \{p_1, p_2, p_4, p_6\}$.

All the nonempty traps are marked by C_{in}, and every nonempty siphon contains a (marked) trap: $\{p_1, p_3, p_5\}$ and $\{p_2, p_4, p_6\}$ are traps themselves, $\{p_1, p_3, p_4, p_5, p_6\}$ contains $\{p_1, p_3, p_5\}$, and $\{p_2, p_3, p_4, p_5, p_6\}$ contains $\{p_2, p_4, p_6\}$. Hence, according to Theorem 212, M is live (note that M has no isolated places).

Since M is live (and has no isolated places), Theorem 213 is applicable. From Exercise 9.9 we know that the sequential components of M are $\{p_1, p_3, p_5\}$ and $\{p_2, p_4, p_6\}$. Since they cover M, M is safe.

We observe that instead of computing the sequential components of M as in Exercise 9.9, we can obtain them as follows. By Lemma 195, sequential components are subsystems (i.e., both siphons and traps). Hence, the only candidate sequential components are $\{p_1, p_3, p_5\}$

and $\{p_2, p_4, p_6\}$. It can be checked with Corollary 193 that (the characteristic functions of) both sets are characteristic p-invariants, and since they have value 1, they are both sequential components of M (see Definition 188).

9.12 a) The siphons are: $\emptyset, P_M, \{p_1, p_3, p_4\}, \{p_1, p_2, p_4\}$.
The traps are: $\emptyset, P_M, \{p_1, p_3, p_4\}, \{p_2, p_4\}, \{p_2, p_3, p_4\}$.

b) By Theorem 212 M is live as P_M and $\{p_1, p_3, p_4\}$ contain the marked trap $\{p_1, p_3, p_4\}$, and $\{p_1, p_2, p_4\}$ contains the marked trap $\{p_2, p_4\}$.
The only subsystem is $\{p_1, p_3, p_4\}$, and it is a sequential component by Corollary 193 (cf. the answer to Exercise 9.11). Thus, p_2 is not covered by any sequential component, and so M is not safe (by Theorem 213).

c) It is not, as the siphon $\{p_1, p_2, p_4\}$ contains only the trap $\{p_2, p_4\}$, which is not marked: $C(p_2) = C(p_4) = 0$.