Solutions of the exercises of Chapters 6 and 7 for

Theorie van Concurrency

6.1 $\mathbf{li}_\rho =$

$$\{(a,a),(a,b),(a,c),(a,d),(a,f),(a,g),$$
$$(b,a),(b,b),(b,c),(b,d),(b,f),(b,g),$$
$$(c,a),(c,b),(c,c),$$
$$(d,a),(d,b),(d,d),(d,f),(d,g),$$
$$(e,e),(e,f),(e,g),$$
$$(f,a),(f,b),(f,d),(f,e),(f,f),(f,g),$$
$$(g,a),(g,b),(g,d),(g,e),(g,f),(g,g)\}$$

See Fig.1.

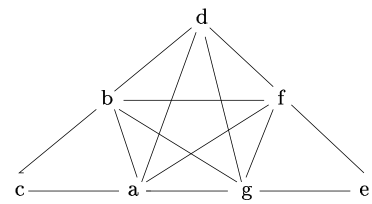

Figure 1: $\mathbf{li}_\rho$ for ex. 6.1

$\mathbf{co}_\rho =$

$$\{(a,a),(a,e),$$
$$(b,b),(b,e),$$
$$(c,c),(c,d),(c,e),(c,f),(c,g),$$
$$(d,c),(d,d),(d,e),$$
$$(e,a),(e,b),(e,c),(e,d),(e,e),$$
$$(f,c),(f,f),$$
$$(g,c),(g,g)\}$$

See Fig.2.

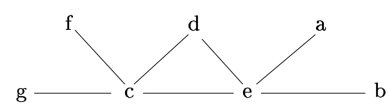

Figure 2: $\mathbf{co}_\rho$ for ex. 6.1

6.2 (*i*) The cliques are:
$\emptyset$ (the empty set),
$\{a\},\{b\},\{c\},\{d\},\{e\},\{f\},\{g\}$ (all singletons),
$\{b,c\},\{b,d\},\{c,e\},\{d,e\},\{d,f\},\{d,g\},\{e,g\},\{f,g\}$ (all undirected edges),
$\{d,e,g\},\{d,f,g\}$ (all "triangles").
The maximal cliques are: $\{a\},\{b,c\},\{b,d\},\{c,e\},\{d,e,g\},\{d,f,g\}$.

(*ii*) The maximal cliques are $\{a,b,d\}$ and $\{b,c,d,e\}$. The cliques are all the subsets of the maximal cliques.

(*iii*) The maximal cliques are $\{a, c\}$, $\{c, e\}$, $\{c, f\}$, and $\{b, c, d\}$.

6.3 Lines of ρ: $\{a, b, c\}, \{a, b, d, f, g\}, \{e, f, g\}$
Cuts of ρ: $\{a, e\}, \{b, e\}, \{c, d, e\}, \{c, f\}, \{c, g\}$
For every line L and every cut C: $L \cap C \neq \emptyset$. So ρ is dense.

6.4 (a) See Fig.3.

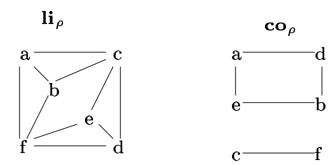

Figure 3: $\mathbf{li}_\rho$ and $\mathbf{co}_\rho$ for ex. 6.4(a)

(b) Lines of ρ: $\{a, b, c\}, \{a, b, f\}, \{c, d, e\}, \{d, e, f\}$
Cuts of ρ: $\{a, d\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, f\}$
ρ is dense

6.5 a) $\{d, f, g\}$ b) $\{a, b, d\}$ c) $\{a\}$ d) $\{a, b, c\}$ e) $\{f\}$ f) $\{b, e\}$

6.6 a) $\{c, e, f\}$ b) $\{a, b, d\}$ c) $\{b, c, d, e, f\}$ d) $\{c, d\}$ e) $\{c, f\}$ f) $\{a, d\}$

6.7 See Fig.4

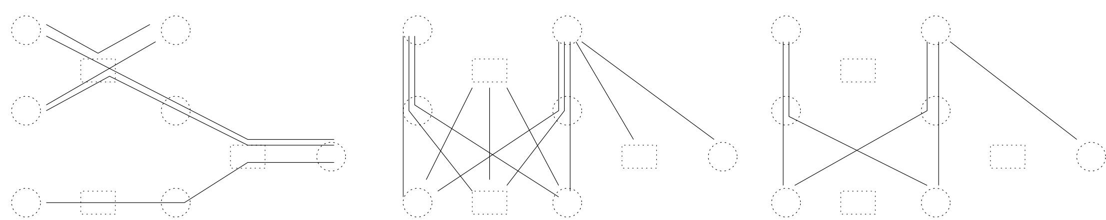

Figure 4: Lines, cuts, and slices for ex. 6.7

- By Theorem 79, the reachable configurations of N are the slices of N.
- By Theorem 83, the sequential components of N are its lines (which are the paths from $^\circ N$ to N°, by Lemma 82 in the Errata).
- It is dense as each line and cut intersect (see Theorem 84).

6.8 Proof of Theorem 71.

(1) See the lecture notes.

(2) The first and the last equation follow immediately from Lemma 70. The other equations follow immediately from the definitions.

(3) To prove that $A \subseteq {}^{\rightarrow}B \cup B^{\rightarrow}$, consider an arbitrary $a \in A$. If $a \in B$ then we are ready because $B \subseteq {}^{\rightarrow}B$ (and $B \subseteq B^{\rightarrow}$). Now assume that $a \notin B$. Since B is a maximal **co**-clique, there exists $b \in B$ such that not $a\,\mathbf{co}\,b$. Thus, either $a\,\rho\,b$ or $b\,\rho\,a$. In the first case $a \in {}^{\rightarrow}B$ and in the second case $a \in B^{\rightarrow}$.

To prove that ${}^{\rightarrow}B \cap B^{\rightarrow} \subseteq B$, consider an arbitrary $a \in {}^{\rightarrow}B \cap B^{\rightarrow}$ and suppose that $a \notin B$. Then there exist b_1, b_2 such that $a\,\rho\,b_1$ and $b_2\,\rho\,a$. Hence, by transitivity, $b_2\,\rho\,b_1$ which contradicts the fact that B is a **co**-clique.

(4) From ${}^{\rightarrow}B \cup B^{\rightarrow} = A$ in (3) it easily follows that $({}^{\rightarrow}B)^{\rightarrow} = A$. It is also easy to show that ${}^{\circ}(C^{\rightarrow}) = {}^{\circ}C$ for every C. Hence, taking $C = {}^{\rightarrow}B$, ${}^{\circ}A = {}^{\circ}(C^{\rightarrow}) = {}^{\circ}C = {}^{\circ}({}^{\rightarrow}B)$. This proves the first equation. The last equation can be shown in the same way. The third equation is proved as follows: ${}^{\circ}(B^{\rightarrow}) = {}^{\circ}B = B$ and similarly for the second equation.

6.9 (a) From Definition 88.

(1) We have to check that N is $(P, \mathbf{use}(T))$-labelled. It is obviously (P, T)-labelled. To check that the labels $t_1, \ldots, t_5$ are in $\mathbf{use}(T)$ we could construct $SCG(M)$ (see Figure 5). However, we do not have to do this because it follows automatically from Theorem 81(1) and Theorem 90 (which also hold for $\Sigma_2 = T$, as can easily be checked).

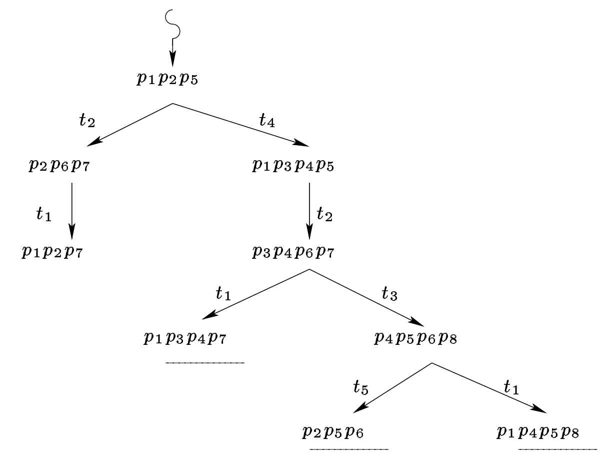

Figure 5: $SCG(M)$ for ex. 6.9(a)

(2,3) $\phi_1({}^{\circ}N) = \{p_1, p_2, p_5\} = C_{in}$, and $\phi_1 \restriction {}^{\circ}N$ is injective because $\#({}^{\circ}N) = \#C_{in} = 3$.

(4,5) Check the transitions of N one by one.

(b) $x = s_1 s_2 s_3 s_4 s_5 s_6$ with $\phi(x) = t_2 t_1 t_4 t_3 t_5 t_4$.
$\phi({}^{\circ}N) = \{p_1, p_2, p_5\}$, $\phi(N^{\circ}) = \{p_1, p_3, p_4, p_5\}$.

(c) See Fig.6.

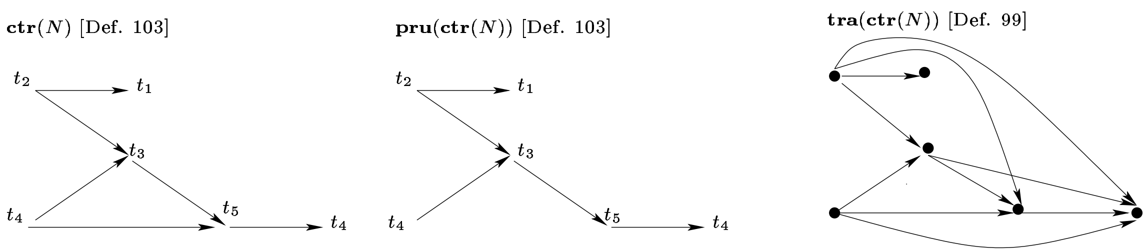

Figure 6: $\mathbf{ctr}(N)$, $\mathbf{pru}(\mathbf{ctr}(N))$ and $\mathbf{tra}(\mathbf{ctr}(N))$ for ex. 6.9(c)

(d) See Fig.7.

6.10 See Figs. 8, 9, and 10.

6.11 (a) [Theorems 59 and 49]
Playing around we get sequential components: $\{p_1, p_2, p_4, p_6, p_{11}, p_8\}$, $\{p_1, p_3, p_5, p_6, p_{11}, p_8\}$, and $\{p_7, p_9, p_{10}, p_{11}\}$ covering M.

(b,c) See Figs. 11 and 12, respectively.

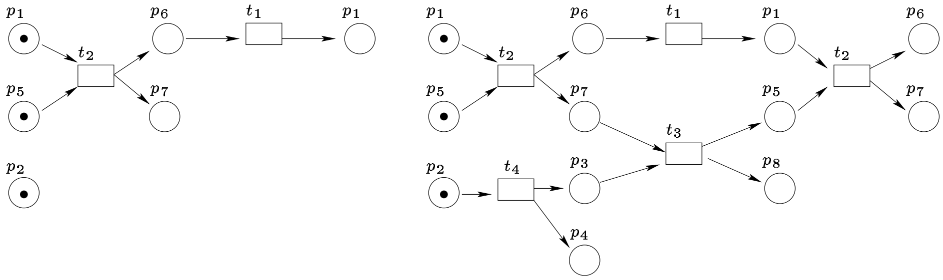

Figure 7: Answer to ex. 6.9(d)

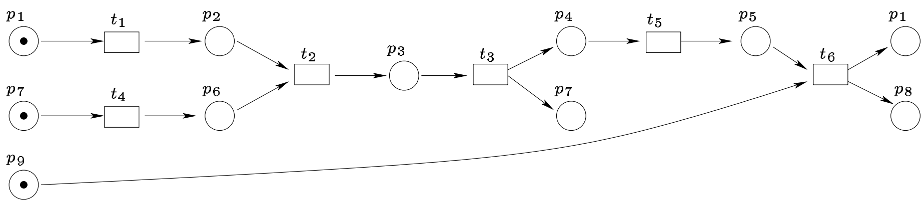

Figure 8: Answer to ex. 6.10(a)

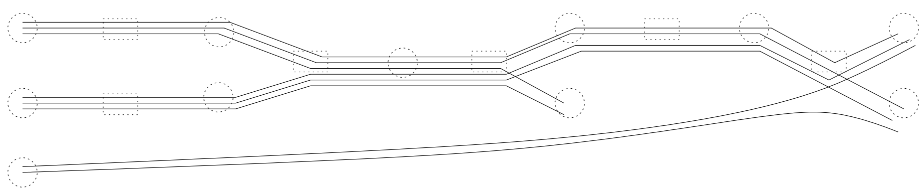

Figure 9: Sequential components for ex. 6.10(b)

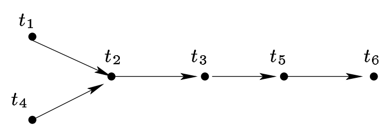

Figure 10: **ctr**(N) [= **pru**(**ctr**(N))] for ex. 6.10(c)

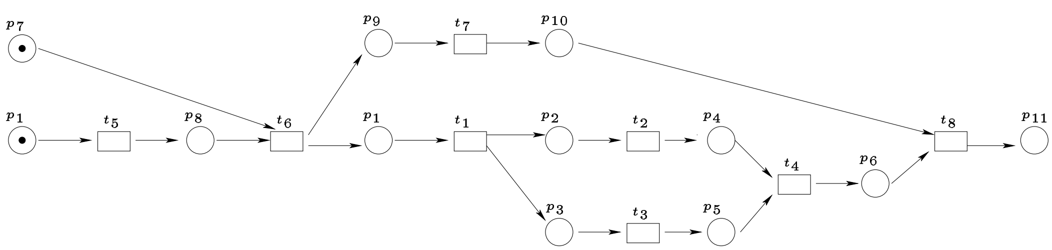

Figure 11: Answer to ex. 6.11(b)

7.1 In what follows we will show that for every acyclic labelled graph G, $\mathbf{top}(\mathbf{tra}(G)) = \mathbf{top}(G)$. From that Lemma 111 follows: if $\mathbf{tra}(G) = \mathbf{tra}(G')$, then $\mathbf{top}(G) = \mathbf{top}(\mathbf{tra}(G)) = \mathbf{top}(\mathbf{tra}(G')) = \mathbf{top}(G')$ and hence also, by the definition of **words**, $\mathbf{words}(G) = \mathbf{words}(G')$. And from this

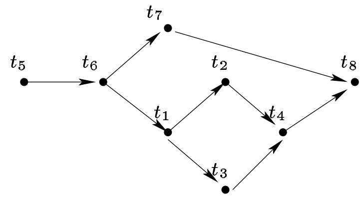

Figure 12: $\mathbf{ctr}(N)$ [$= \mathbf{pru}(\mathbf{ctr}(N))$] for ex. 6.11(c)

lemma the statement of the exercise follows because $\mathbf{tra}(\mathbf{pru}(G)) = \mathbf{tra}(G)$ (by Theorem 101) and $\mathbf{tra}(\mathbf{tra}(G)) = \mathbf{tra}(G)$ (obviously).

To prove that $\mathbf{top}(\mathbf{tra}(G)) = \mathbf{top}(G)$, we have to show that $u_1 \cdots u_n$ is a topological order of $\mathbf{tra}(G)$ if and only if it is a topological order of G (where all u_i are distinct and $V = \{u_1, \ldots, u_n\}$). In other words, we have to show:

for all $1 \leq i, j \leq n$, $(u_i, u_j) \in \Gamma^+$ implies $i < j$
iff
for all $1 \leq i, j \leq n$, $(u_i, u_j) \in \Gamma$ implies $i < j$.

This is obvious for the only-if direction, because $\Gamma \subseteq \Gamma^+$. To show the if direction, assume that $(u_i, u_j) \in \Gamma$ implies $i < j$ for all i and j, and assume that $(u_i, u_j) \in \Gamma^+$. Then there exist $1 \leq k_1, \ldots, k_m \leq n$ such that $k_1 = i$, $k_m = j$ and $(u_{k_r}, u_{k_{r+1}}) \in \Gamma$ for all $1 \leq r < m$. Hence, by the first assumption, $k_r < k_{r+1}$ for all $1 \leq r < m$. So, $k_1 < k_m$, i.e., $i < j$.

7.2 a) N is depicted in Fig.11 and $\mathbf{ctr}(N) = \mathbf{pru}(\mathbf{ctr}(N))$ is depicted in Fig.12.
$\mathbf{words}(\mathbf{pru}(\mathbf{ctr}(N)))$ has the following elements:
$t_5t_6t_7t_1t_2t_3t_4t_8, t_5t_6t_7t_1t_3t_2t_4t_8,$
$t_5t_6t_1t_7t_2t_3t_4t_8, t_5t_6t_1t_7t_3t_2t_4t_8,$
$t_5t_6t_1t_2t_7t_3t_4t_8, t_5t_6t_1t_3t_7t_2t_4t_8,$
$t_5t_6t_1t_2t_3t_7t_4t_8, t_5t_6t_1t_3t_2t_7t_4t_8,$
$t_5t_6t_1t_2t_3t_4t_7t_8, t_5t_6t_1t_3t_2t_4t_7t_8.$
Let us check that $t_5t_6t_1t_2t_3t_4t_7t_8$ is a firing sequence of M:
$C_{in} = \{p_1, p_7\}[t_5\rangle\{p_7, p_8\}[t_6\rangle\{p_1, p_9\}[t_1\rangle\{p_2, p_3, p_9\}[t_2\rangle\{p_3, p_4, p_9\}[t_3\rangle\{p_4, p_5, p_9\}[t_4\rangle$
$\{p_6, p_9\}[t_7\rangle\{p_6, p_{10}\}[t_8\rangle\{p_{11}\}.$

b) See the proof of Theorem 114. Construct the process $N \in \mathrm{PROC}(M)$ corresponding to $t_1t_2t_3t_4$ according to Theorem 92, and construct $G = \mathbf{pru}(\mathbf{ctr}(N))$. See Fig.13. Then $G \in \mathrm{LPO}(M)$ and $t_1t_2t_3t_4 \in \mathbf{words}(G) = \{t_1t_2t_3t_4, t_1t_3t_2t_4\}$.

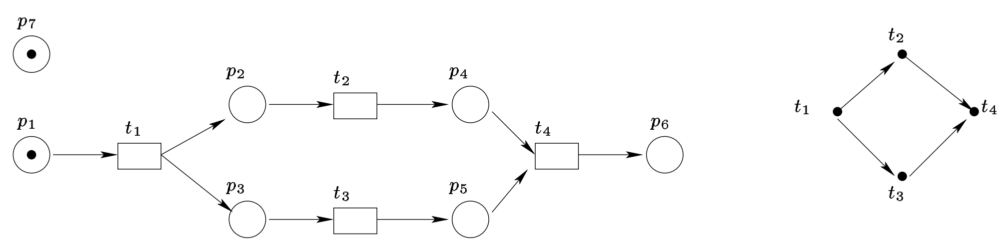

Figure 13: Process N and $G = \mathbf{pru}(\mathbf{ctr}(N)) \in \mathrm{LPO}(M)$ for ex. 7.2(b)

7.3 a) $\mathrm{SCG}(M)$ is given in Fig.14. From the diamonds in $\mathrm{SCG}(M)$ we can see which concurrent steps are possible.
So $\mathbf{ind}(M) = \{(t_1, t_2), (t_2, t_1), (t_1, t_5), (t_5, t_1), (t_4, t_2), (t_2, t_4), (t_4, t_5), (t_5, t_4)\}$.

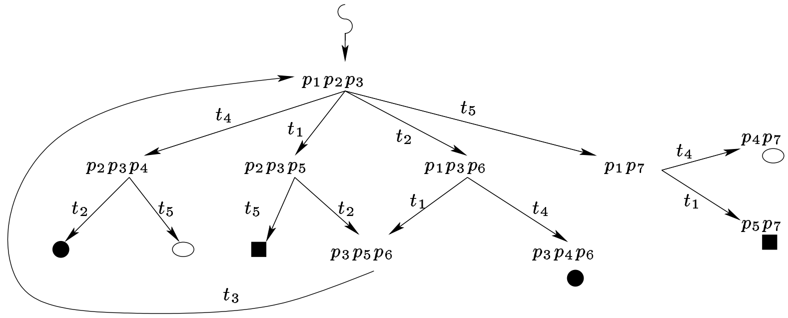

Figure 14: SCG(M) for ex. 7.3(a)

b) $\mathbf{dep}(M) = \{(t_1, t_4), (t_4, t_1), (t_2, t_5), (t_5, t_2)\} \cup \{(t_3, t_i), (t_i, t_3) \mid i \in \{1, 2, 4, 5\}\}$ $\cup\{(t_i, t_i) \mid 1 \leq i \leq 5\}$ (note that M is reduced).
$\mathbf{dep}_M(t_1t_2t_3t_4t_2)$ and $\mathbf{pru}(\mathbf{dep}_M(t_1t_2t_3t_4t_2))$ are depicted in Fig.15.

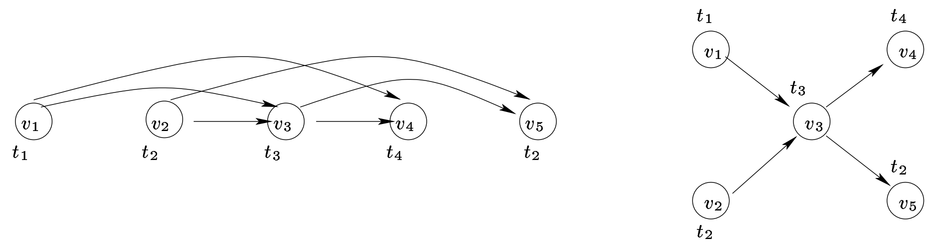

Figure 15: $\mathbf{dep}_M(x)$ and $\mathbf{pru}(\mathbf{dep}_M(x))$ for ex. 7.3(b)

c) See Fig.16.

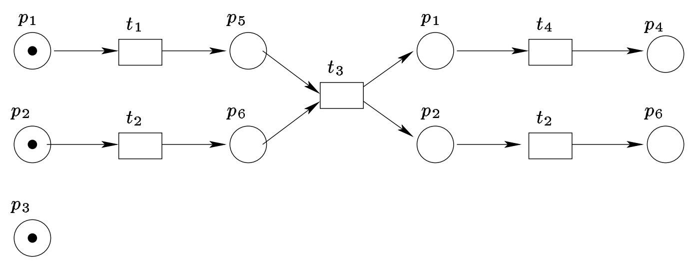

Figure 16: Process N for ex. 7.3(c)

d) Additional exercise: Determine the trace $[t_1t_2t_3t_4t_2]_{\mathbf{ind}(M)}$. Solution: By Theorem 134, $[t_1t_2t_3t_4t_2]_{\mathbf{ind}(M)} = \mathbf{words}(\mathbf{dep}_M(t_1t_2t_3t_4t_2))$ which equals $\mathbf{words}(\mathbf{pru}(\mathbf{dep}_M(t_1t_2t_3t_4t_2)))$. By b), see Fig.15, this is the set $\{t_1t_2t_3t_4t_2, t_1t_2t_3t_2t_4, t_2t_1t_3t_4t_2, t_2t_1t_3t_2t_4\}$.

7.4 By Theorem 92, $\text{FS}(M) = \{\phi(s_1)\cdots\phi(s_n) \mid {}^\circ N[s_1\cdots s_n\rangle N^\circ, N \in \text{PROC}(M)\}$. Hence $\{\mathbf{pru}(\mathbf{dep}_M(x)) \mid x \in \text{FS}(M)\} =$
$\{\mathbf{pru}(\mathbf{dep}_M(\phi(s_1)\cdots\phi(s_n))) \mid {}^\circ N[s_1\cdots s_n\rangle N^\circ, N \in \text{PROC}(M)\}$.
From Theorem 79 or Theorem 112 we know that for every process net N there exists at least one firing sequence ${}^\circ N[s_1\cdots s_n\rangle N^\circ$. Hence, Theorem 122 implies that
$\{\mathbf{pru}(\mathbf{dep}_M(\phi(s_1)\cdots\phi(s_n))) \mid {}^\circ N[s_1\cdots s_n\rangle N^\circ, N \in \text{PROC}(M)\} \equiv_\beta$
$\{\mathbf{pru}(\mathbf{ctr}(N)) \mid N \in \text{PROC}(M)\}$, which is LPO$(M)$.

7.5 The purpose of this exercise is not very clear. From the solution to Exercise 5.1 we know that M is firing sequence equivalent with M_1 and M_2, but not with M_3. By Theorem 126 the same holds for lpo-equivalence.

7.6 (a) By Definition 131, $[bcda]_I = \{bcda, cbda, cdba, cdab, dcab, dcba, dbca, bdca\}$. See also Fig.17 where each line represents $\doteq$.

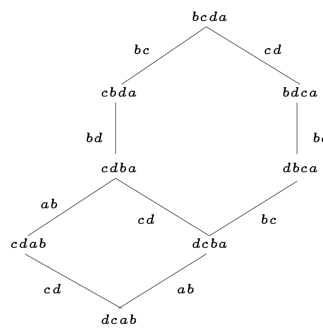

Figure 17: $[bcda]_I$ for ex. 7.6(a)

(b) See Fig.18.

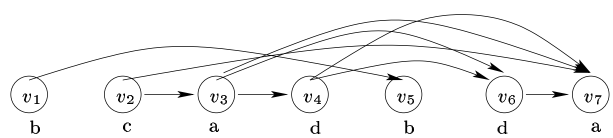

Figure 18: $\mathbf{dep}_I(bcadbda)$ for ex. 7.6(b)

(c) See Fig.19.

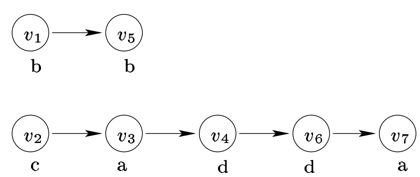

Figure 19: $\mathbf{pru}(\mathbf{dep}_I(bcadbda))$ for ex. 7.6(c)

(d) The words in $\mathbf{words}(\mathbf{dep}_I(bcadbda))$ are obtained by shuffling the words bb and $cadda$ in all possible ways. This gives the following 21 words:
bbcadda, *cbbadda*, *cabbdda*, *cadbbda*, *caddbba*, *caddabb*,
bcbadda, *cbabdda*, *cabdbda*, *cadbdba*, *caddbab*,
bcabdda, *cbadbda*, *cabddba*, *cadbdab*,
bcadbda, *cbaddba*, *cabddab*,
bcaddba, *cbaddab*,
bcaddab.
Note that this is $[bcadbda]_I$, see Theorem 134.

7.7 (a) Following the style of Exercise 7.6 we get Fig.20, which means that $[abecd]_I = \{abecd, abced, acbed, acebd, aecbd, aebcd, aebdc, abedc, baedc, baecd, baced\}$.

(b) See Fig.21.

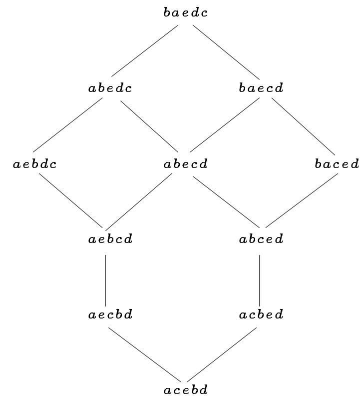

Figure 20: $[abecd]_I$ for ex. 7.7(a)

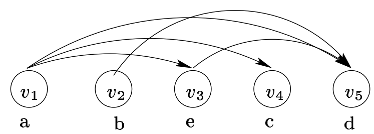

Figure 21: $\mathbf{dep}_I(abecd)$ for ex. 7.7(b)

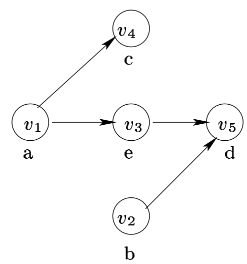

Figure 22: $\mathbf{pru}(\mathbf{dep}_I(abecd))$ for ex. 7.7(c)

(c) See Fig.22.

(d) $\mathbf{words}(\mathbf{pru}(\mathbf{dep}_I(abecd)))$ = [Lemma 111] = $\mathbf{words}(\mathbf{dep}_I(abecd))$ = [Theorem 134] = $[abecd]_I$

7.8 (a) $\mathbf{ind}(M) = \{(t_2, t_4), (t_4, t_2), (t_3, t_4), (t_4, t_3)\}$. See also the solution of Exercise 5.9(b).

(b) See Fig.23.

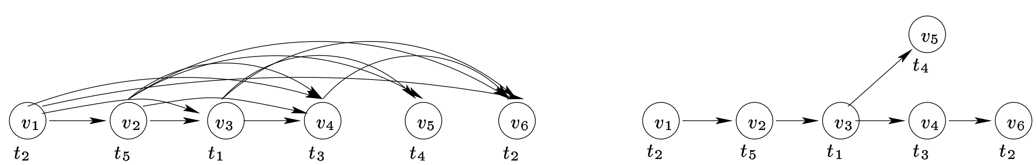

Figure 23: $\mathbf{dep}_M(x)$ and $\mathbf{pru}(\mathbf{dep}_M(x))$ for ex. 7.8(b)

(c) $[t_2t_5t_1t_3t_4t_2]_{\mathbf{ind}(M)} = \{t_2t_5t_1t_3t_4t_2, t_2t_5t_1t_4t_3t_2, t_2t_5t_1t_3t_2t_4\}$.

(d) Additional exercise: Give a process N of M' (see Exercise 5.9(e)) for which $\mathbf{pru}(\mathbf{ctr}(N)) \equiv \mathbf{pru}(\mathbf{dep}_M(x))$. Solution: see Fig.24.

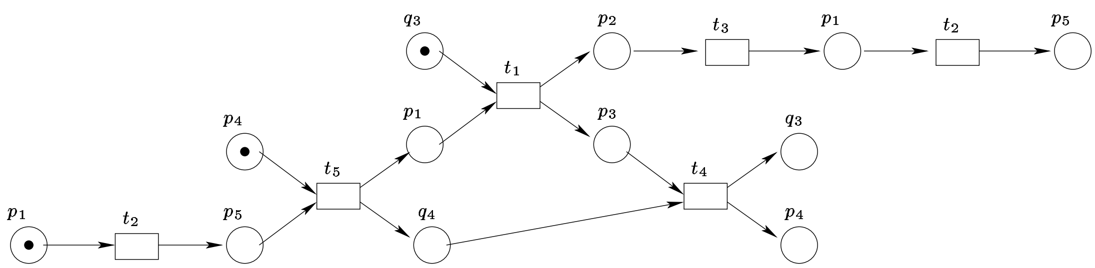

Figure 24: Process N with $\mathbf{pru}(\mathbf{ctr}(N)) \equiv \mathbf{pru}(\mathbf{dep}_M(x))$ for ex. 7.8(d)