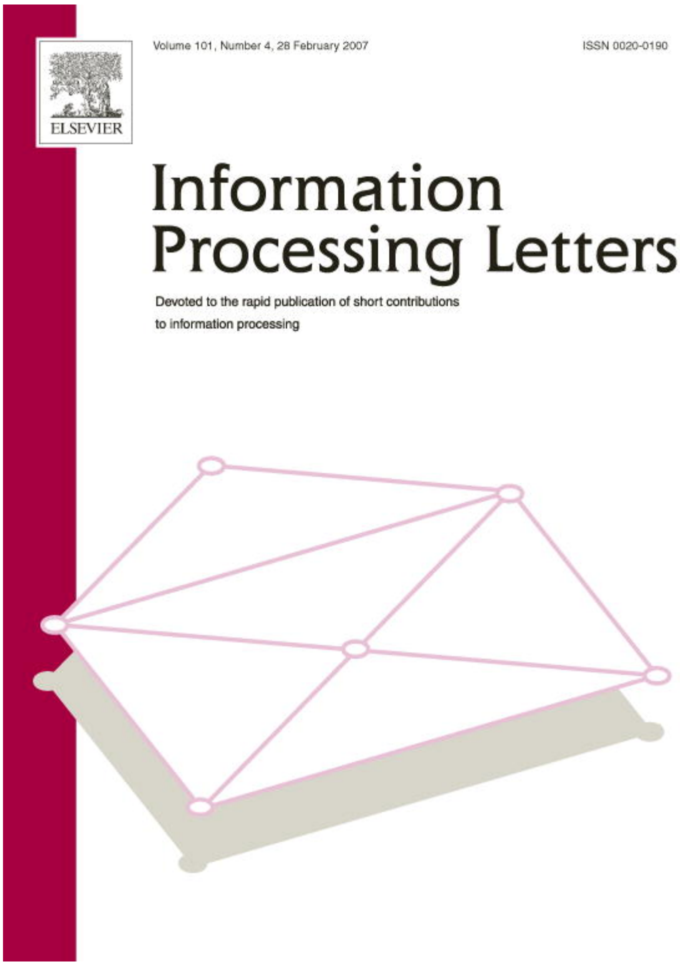
Volume 101, Number 4, 28 February 2007
ISSN 0020-0190
ELSEVIER
Information Processing Letters
Devoted to the rapid publication of short contributions to information processing

Available online at www.sciencedirect.com
ScienceDirect
Information Processing Letters 101 (2007) 139–140
Information Processing Letters
www.elsevier.com/locate/ipl

A Kleene characterization of computability

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Received 25 August 2006
Available online 9 October 2006
Communicated by L. Boasson

Abstract

The recursively enumerable binary relations are the smallest class containing all singleton relations and closed under union, concatenation, Kleene star, composition, and transitive closure.

Keywords: Formal languages; Kleene; Recursive enumerable; Binary relation

There are many characterizations of computability, but the one presented here does not seem to appear explicitly in the literature. Nevertheless, it is a natural and simple characterization, very similar to Kleene's characterization of the regular languages as the smallest class of languages containing all finite languages and closed under union, concatenation, and Kleene star. As such it deserves more attention. Traditionally, the theory of computability centers around the notions of a (partially) recursive function and of a recursively enumerable set. It is well known, and easy to see, that a partial function f is (partially) recursive if and only if the set $\{(x, y) \mid y = f(x)\}$ is recursively enumerable. Also, obviously, a set A is recursively enumerable if and only if the set $\{(x, x) \mid x \in A\}$ is recursively enumerable. Since both these sets are binary relations, it therefore suffices to give a characterization of the recursively enumerable binary relations.

We consider binary relations R for which there exists an alphabet Σ with $R \subseteq \Sigma^* \times \Sigma^*$ (i.e., R is a binary relation on the set of strings over alphabet Σ). The relevant operations on such string relations are the following. First, the usual set-theoretic operations of union, composition, and transitive closure: $R \cup S = \{(x, y) \mid (x, y) \in R \text{ or } (x, y) \in S\}$, $R \circ S = \{(x, z) \mid \exists y\colon (x, y) \in R, (y, z) \in S\}$ (note the order), and $R^+ = \bigcup_{n=1}^{\infty} R^n$ with $R^1 = R$ and $R^{n+1} = R \circ R^n$. Second, (coordinate-wise) concatenation and Kleene star: $R \cdot S = \{(x_1x_2, y_1y_2) \mid (x_1, y_1) \in R, (x_2, y_2) \in S\}$ and $R^\star = \bigcup_{n=0}^{\infty} R^n$ with, this time, $R^0 = \{(\lambda, \lambda)\}$ (where λ denotes the empty string) and $R^{n+1} = R \cdot R^n$. Note that composition combines relations in a sequential way (in time), whereas concatenation combines them in a parallel way (in space). Moreover, transitive closure and Kleene star are the iteration of composition and concatenation, respectively. This makes our characterization into a kind of "double" Kleene result.

Let us denote by RER the smallest class of binary relations that contains the empty relation $\emptyset$ and all singleton binary relations (i.e., all relations $\{(u, v)\}$ where u and v are strings over some alphabet), and is closed under the above five operations. We now show that RER is the class of all recursively enumerable binary rela-

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doi:10.1016/j.ipl.2006.09.003

tions. In one direction this is obvious: singleton relations are recursively enumerable, and the five operations preserve recursive enumerability (an easy exercise). In the other direction we proceed as follows.

A *rewriting system* (or semi-Thue system, or Chomsky type 0 grammar) is a finite binary relation $P \subseteq \Delta^* \times \Delta^*$ (for some alphabet Δ). The *derivation relation* of the rewriting system is defined as usual: for strings $x, y \in \Delta^*$, $x \Rightarrow_P y$ if and only if there exist $(u, v) \in P$ and $x_1, x_2 \in \Delta^*$ such that $x = x_1 u x_2$ and $y = x_1 v x_2$. It is a straightforward exercise (for the reader familiar with elementary theory of computation) to show that a binary relation R on Σ^* is recursively enumerable if and only if there exists a rewriting system P on some Δ^* such that $R = \{(x, y) \in \Sigma^* \times \Sigma^* \mid \#x\# \Rightarrow_P^+ y\}$, where # is a distinguished symbol that is in Δ but not in Σ.

Since a nonempty finite set is the union of singletons, RER contains all finite relations (i.e., all rewriting systems). Hence, for an alphabet Δ, RER contains $\{(x, x) \mid x \in \Delta^*\}$ (the identity on Δ^*) because it equals $\mathrm{id}_\Delta^\star$ where id_Δ is the rewriting system $\{(a, a) \mid a \in \Delta\}$. And so, for any rewriting system P on Δ^*, RER contains both the derivation relation $\Rightarrow_P$ which equals $\mathrm{id}_\Delta^\star \cdot P \cdot \mathrm{id}_\Delta^\star$, and its transitive closure $\Rightarrow_P^+$. Consequently, it also contains the recursively enumerable binary relation $R = \{(x, y) \in \Sigma^* \times \Sigma^* \mid \#x\# \Rightarrow_P^+ y\} = \mathrm{in}_\Sigma \circ (\Rightarrow_P)^+ \circ \mathrm{out}_\Sigma$ with $\mathrm{in}_\Sigma = \{(\lambda, \#)\} \cdot \mathrm{id}_\Sigma^\star \cdot \{(\lambda, \#)\}$ and $\mathrm{out}_\Sigma = \mathrm{id}_\Sigma^\star$. This proves that RER is the class of recursively enumerable relations.

It is well known that the smallest class of relations containing the empty set and all singleton relations, and closed under the operations of union, concatenation, and Kleene star, equals the class of rational transductions, i.e., the class of relations that can be computed by finite-state transducers (see Theorem X.3.2 of [3], or Theorem III.6.1 of [2]). It follows from the above proof that RER is the closure of the class of rational transductions under the operations of composition and transitive closure (because $\Rightarrow_P$, in_Σ, and out_Σ are rational transductions). The closure of the class of *length-preserving* rational transductions under the operations of composition and transitive closure is studied in [7] (where transitive closure is called iteration): it equals the class of relations that can be computed by nondeterministic Turing machines in linear space. It is well known that iterated rational transductions can be used to generate the recursively enumerable languages, see, e.g., [8,1]. Characterizations of the recursively enumerable languages form an active field of investigation, see, e.g., [5,6,4].

It also follows from the above proof that RER is the smallest class of relations containing all *finite* relations and closed under the operations of concatenation, Kleene star, composition, and transitive closure (because union was only needed to obtain the finite relations from the singletons). In the same vein (but in the opposite direction), RER is the smallest class of relations closed under the five operations and containing all singleton relations of the form $\{(\lambda, a)\}$ or $\{(a, \lambda)\}$, where a is a symbol. That is because $\emptyset = \{(\lambda, a)\} \circ \{(b, \lambda)\}$ with $a \neq b$, $\{(u, v)\} = \{(u, \lambda)\} \circ \{(\lambda, v)\}$, $\{(uv, \lambda)\} = \{(u, \lambda)\} \cdot \{(v, \lambda)\}$ and similarly for $\{(\lambda, uv)\}$, and $\{(\lambda, \lambda)\} = \{(\lambda, a)\} \circ \{(a, \lambda)\}$.

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