Assignment 1: Multicriteria Optimization and Decision Making

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September 11, 2007

Assignments can be done in pairs or alone. Submit assignments until the lecture on 25th of October. You may hand them in before the lecture or submit online to emmerich@liacs.nl.

1 Multicriteria Comparisons of Products

Washing machines are given Eurobell grades for *washing performance* (WP), e.g. the cleanness of laundry after washing, *energy efficiency* (EE), and *spin efficiency* (SE), measuring the percentage of water that stays in the laundry after centrifugation. *Grades* run from A to E (best to worst). Here an AAA machine will denote top performance on the mention three categories. Note, that recently the additional grades A+ and A++ have been introduced to indicate performances superior to A.

As a customer, one may consider further criteria when buying a washing machine. Obviously the *price* (in Euro) is an important criterion. Moreover, the weight-capacity (in kg), the user-friendliness (+: high, ○: average, -: low), the noise-level (low, medium, high) are interesting criteria. Moreover, one may find the new *speech option* attractive, that allows the user to talk with his/her washing machine.

	WP	EE	SE	Speech	Noise	UF	Weight	Risk	Price
Hermine	A	B	B	Yes	Med	?	6kg	Med	900
AlbertRijn	A	B	A	No	Med	+	6kg	Low	700
Woed/AGE	A	A	A++	No	Very High	○	7kg	Low	800
GreenClean	B	A+	B	No	Low	○	6kg	Med	800
Aldine	A	B	B	No	Med	+	7kg	Med	500
HappyOne	C	C	C	No	High	+	4kg	High	600
WishyWashy	C	C	C	No	High	-	7kg	High	600
Shineandclean	A++	C	B	No	High	-	7kg	High	600
DryandDirty	D	C	A++	No	High	-	7kg	High	600
Talkmaster	D	D	D	Yes	Very High	+	5kg	High	900

Q1.1: Formulate three objectives, and two constraints on the given criteria! Use the standard formulation of mathematical programming and make sure that at least seven solutions stay feasible w.r.t. the constraints!

Q1.2: Draw the parallel coordinate diagram for all feasible solutions w.r.t. your choice of constraints! (to keep it simple, display only the coordinates for the three objectives and discard infeasible solutions)

Q1.3: Determine the non-dominated solutions w.r.t. the Pareto order and the chosen objectives and constraints!

Q1.4: Discuss, whether there are good compromise solutions or not by looking at the trade-offs among different solutions?

As a second example, consider you have to compare **cars** from a representative sample of cars. At the address `http://davis.wpi.edu/xmdv/downloadxmdv.html` a software XMDV for multi-attribute comparisons can be downloaded (alpha6.0).

Q1.5: From the examples[1] load the 'car' example. Determine the subset of cars that have a MPG (miles per gallon) value of less than 30 and at most 4 cylinders. Use the brushing option in the parallel coordinates diagram to identify the feasible subset of solutions (include screen-shot in document).

[1] they are in the data directory of the installation path.

Q1.6: Use the scatterplot-matrix tool of the xmdv Program to answer the following questions: What is the reason for weight reduction in cars? How is the weight correlated with the power (PS) of the cars?

2 Partial orders on numbers

An ordering is defined on the set $X = \{2, 3, \dots, 20\}$ via

$$\forall x, y \in X : x \preceq y :\Leftrightarrow x \text{ modulo } y \equiv 0.$$

Q2.1: Show, whether this order fulfils the transitivity, antisymmetry, and reflexivity axioms or not! Classify the type of ordering, i.e. is it a pre-ordered set, a partially ordered set, or even a linearly ordered set?

Q2.2: Draw the Hasse Diagramm of this ordered set

Q2.3: What are the minimal and maximal elements of the ordered set?

3 Interval orders

In practical optimization, such as business planning of energy management, it is often difficult to predict the behavior of a system in a precise way. However, it is often possible to provide an upper bound and a lower bound for the systems behavior. Based on such bounds we can compare the performance of different system alternatives among which we may choose. Given a set of intervals $Y_1 = [l^1, u^1] \subset \mathbb{R}, \dots, Y^m = [l^m, u^m] \subset \mathbb{R}$, an order can be defined as follows:

$$\forall i, j \in \{1, \dots, m\} : Y^i \text{ possibly dominates } Y^j \text{ (in symbols } Y^i <_? \; Y^j\text{), iff } l^i < u_j \text{ and}$$

$$\forall i, j \in \{1, \dots, m\} : Y^i \text{ certainly dominates } Y^j \text{ (in symbols } Y^i <_! \; Y^j\text{), iff } u^i < l_j$$

In order to gain a deeper insight into ordered sets defined by these *possibilistic* comparisons, we may ask for their axiomatic properties:

Q3.1: Show, whether the $<_!$ and $<_?$ are transitive, (irreflexive, asymmetric) binary relations.

Q3.2: Discuss, whether there exists an order preserving map from $<_!$ to $<_?$ (or an order preserving map from $<_?$ to $<_!$ or not.

Now, consider the Pareto order on a set of interval boxes $\mathbf{Y}^i = [\mathbf{l}^i, \mathbf{u}^i] \subset \mathbb{R}^2, i = 1, \dots, m$, where $\mathbf{l}^i \in \mathbb{R}^2$ define lower bound vectors, and $\mathbf{u}^i \in \mathbb{R}^2$ define upper bound vectors.

$$\mathbf{y} \text{ Pareto dominates } \mathbf{y}' \text{ (in symbols:} \mathbf{y} \prec \mathbf{y}'\text{) :} \tag{1}$$

$$\mathbf{y} \neq \mathbf{y}' \wedge \forall i \in \{1, \dots, n\} : y_i \leq y'_i \tag{2}$$

Based on this definition, introduce possibilistic Pareto domination as follows:

$$\forall i, j \in \{1, \dots, m\} : \mathbf{Y}^i \text{ possibly dominates } \mathbf{Y}^j \text{ (in symbols } \mathbf{Y}^i \prec_? \; \mathbf{Y}^j\text{), iff } \mathbf{l}^i \prec \mathbf{u}^j \text{ and}$$

$$\forall i, j \in \{1, \dots, m\} : \mathbf{Y}^i \text{ certainly dominates } \mathbf{Y}^j \text{ (in symbols } \mathbf{Y}^i \prec_! \; \mathbf{Y}^j\text{), iff } \mathbf{u}^i \prec \mathbf{l}^j$$

Given a set

$$X = \{[\binom{1}{1}, \binom{2}{2}], [\binom{3}{0.5}, \binom{4}{1.5}], [\binom{1.5}{1.5}, \binom{3}{3}], [\binom{2.5}{2.5}, \binom{3.5}{3.5}], [\binom{2}{4}, \binom{4}{4}]\}$$

Q3.3: Visualize the interval boxes in a 2-D diagram!

Q3.4: Draw the graph for the binary relation $\prec_?$ and $\prec_!$ and determine minimal elements of the interval ordered sets! Why can the Hasse diagram not be used for visualizing the interval ordered set $\prec_?$?

Q3.5 Characterize the ordered set geometrically using set inclusion by cones! Be as concise as possible!

4 Pareto orders and constraints

Given an optimization task: $f_1(x) \to \min, \dots, f_n(x) \to \min$, $g_1(x) \leq 0, \dots, g_m(x) \leq 0$ $x \in X$ and let X_E denote the efficient set for this problem.

Q4.1: Which of the following statements holds, if a further objective function is added to the set of objectives? Give reasons for your answer!

A The size of the new efficient set does not decrease.

B The size of the efficient set does not increase.

C The size may either increase or decrease.

Q4.2: Which of the following statements holds, if a further constraint function is added to the set of constraint functions? Give reasons for your answer!

A The size of the new efficient set does not decrease.

B The size of the efficient set does not increase.

C The size may either increase or decrease.