Selecting Patches of Land (6 Points)

1. Formulate the problem of finding a selection of exactly 5 land patches. Maximize the ecological value and make sure that the total price is smaller than or equal to €15000.

\[-\sum_{i=1}^{3} \sum_{j=1}^{5} e_{i,j} x_{i,j} \rightarrow \min\]

subject to

\[-\sum_{i=1}^{3} \sum_{j=1}^{5} p_{i,j} x_{i,j} + 15000 \geq 0\]

\[x_{i,j} \geq 0 \quad \text{for } (i, j) \in \{1, 2, 3\} \times \{1, 2, 3, 4, 5\}\]

\[1 - x_{i,j} \geq 0\]

\[\sum_{i=1}^{3} \sum_{j=1}^{5} x_{i,j} - 5 = 0\]

\[x_{i,j} \in \mathbb{Z} \quad \text{for } (i, j) \in \{1, 2, 3\} \times \{1, 2, 3, 4, 5\}\]
Selecting Patches of Land

2. Formulate the problem of finding a selection of land patches. Maximize the ecological value and make sure that the total price is smaller than or equal to €15000. Make sure that either patch (1, 2) or patch (2, 5) is included.

\[- \sum_{i=1}^{3} \sum_{j=1}^{5} e_{i,j} x_{i,j} \rightarrow \min\]

subject to

\[- \sum_{i=1}^{3} \sum_{j=1}^{5} p_{i,j} x_{i,j} + 15000 \geq 0\]

\[x_{i,j} \geq 0\]

\[1 - x_{i,j} \geq 0\]

\[\sum_{i=1}^{3} \sum_{j=1}^{5} x_{i,j} - 5 = 0\]

\[x_{1,2} + x_{2,5} = 1\]

\[x_{i,j} \in \mathbb{Z} \quad \text{for} \ (i, j) \in \{1, 2, 3\} \times \{1, 2, 3, 4, 5\}\]
Selecting connected patches (1/2)

3. Difficult: Formulate the problem of finding a selection of 5 connected land patches with maximal ecological value. Two patches are connected if they have a common border line segment (not only one point). Hint: One way of for solving this problem is to introduce a adjacency matrix for the neighborhood graph. For a pair of nodes, say \((k, l)\) and \((u, v)\) the constant \(w_{(k, l);(u, v)}\) is 1 if the nodes are connected, otherwise \(w_{(k, l);(u, v)}\) obtains a larger value. Such weights could be used as constants in the formulation of the problem. The fact that two land patches are selected can be indicated by the product of the corresponding binary decision variables being one.

We define constants \(w_{(i,j);(j,k)} = \begin{cases} 1 & \text{if cell } i, j \text{ is direct neighbor of cell } k, \ell \\ w^+ & \text{otherwise} \end{cases} \)

for some \(w^+ > 1\)
Selecting connected patches (2/2)

\[- \sum_{i=1}^{3} \sum_{j=1}^{5} e_{i,j} x_{i,j} \rightarrow \min \quad \text{subject to}\]

There are in total at least 4x2 directed arcs between selected cells

\[(225w^+ - 8(w^+ - 1)) \ldots\]

\[- \sum_{i=1}^{3} \sum_{j=1}^{5} \sum_{k=1}^{3} \sum_{\ell=1}^{5} \left( w(i,j),(k,\ell) x_{i,j} x_{k,\ell} w(i,j),(k,\ell) + (1 - x_{i,j} x_{k,\ell}) w^+ \right) \geq 0 \]

There is no selected node that has no selected neigbour

\[15w^+ - \sum_{k=1}^{3} \sum_{\ell=1}^{5} x(i,j) w(i,j),(k,\ell) \geq 0 \quad \text{for} \quad i = 1, \ldots, 3, \quad j = 1, \ldots, 5\]

In total 5 cells are selected

\[\sum_{i=1}^{3} \sum_{j=1}^{5} x_{i,j} - 5 = 0\]

\[x_{i,j} \geq 0 \quad 1 - x_{i,j} \geq 0 \quad \text{for} \quad (i, j) \in \{1, 2, 3\} \times \{1, 2, 3, 4, 5\}\]

\[x_{i,j} \in \mathbb{Z} \quad \text{for} \quad (i, j) \in \{1, 2, 3\} \times \{1, 2, 3, 4, 5\}\]

It is also possible to define other weight functions (e.g. 0 arc exists, 1 arc does not exist) and proceed in a similar way...
Cutting cookies from a dough (4 Points)

\[- \sum_{i=1}^{100} b_i \rightarrow \min\]

subject to

\[b_i b_j (2R - \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}) \geq 0 \text{ for } (i, j) \in \{1, \ldots, 100\}^2 \setminus \{(1, 1), \ldots, (100, 100)\}.\]

\[1 - b_i \geq 1\]

\[b_i \geq 0\]

\[x_i \geq R, \ y_i \geq R,\]

\[x_i \cdot a - R, \ y_i \cdot b - R, \text{ for } i = 1, \ldots, 100\]

\[b_i \in \mathbb{Z}, \ x_i, y_i \in \mathbb{R} \text{ for } i = 1, \ldots, 100\]

This is a mixed integer non-linear programming problem (MINLP)
Consider your task is to assemble a food package for people in a refugee camp and your monetary budget is limited. The following restrictions apply: Assume that each person needs at least 2500Kcal per day, 60g fat per day, 220g carbs per day, 90g protein per day, and 9mg Vitamin C per day.

In Table 2 food data (per 100g of food) is provided:\[1:\]

1. Formulate the problem of making a cost minimal food package that fulfills the nutrient requirements as a mathematical programming task assuming that the amount of food per package can be chosen on a continuous scale.

2. Solve the mathematical program using a solver of your choice, for instance the free online solver http://apmonitor.com/ or LP_Solve.

3. What happens to the solution, if butter ($j = 5$) is discarded as a food type? Discuss briefly whether the obtained solution makes sense from a practical point of view.

4. Now formulate the problem for the case that only a discrete number of servings (portions) can be put in the packages. Use the following data (in g/serving): 1 slice of bread 50g, 1 orange 200g, 1 fish 500g, 1 date 10g, 1 small package of butter 8g. Compute the number of servings.

<table>
<thead>
<tr>
<th>Type</th>
<th>Energy KCal/100g</th>
<th>Fat g/100g</th>
<th>Carbs g/100g</th>
<th>Protein g/100g</th>
<th>VitC mg/100g</th>
<th>Price Euro/kg</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bread</td>
<td>266</td>
<td>3.29</td>
<td>50.61</td>
<td>7.64</td>
<td>0</td>
<td>1.2</td>
</tr>
<tr>
<td>Fishes</td>
<td>96</td>
<td>1.7</td>
<td>0</td>
<td>20.08</td>
<td>0</td>
<td>3.5</td>
</tr>
<tr>
<td>Oranges</td>
<td>47</td>
<td>0.12</td>
<td>11.75</td>
<td>0.94</td>
<td>9</td>
<td>0.6</td>
</tr>
<tr>
<td>Dates</td>
<td>282</td>
<td>0.39</td>
<td>75.03</td>
<td>2.45</td>
<td>0.1</td>
<td>6.2</td>
</tr>
<tr>
<td>Butter</td>
<td>717</td>
<td>81</td>
<td>0.06</td>
<td>0.85</td>
<td>0</td>
<td>5.2</td>
</tr>
</tbody>
</table>

Table 1: Data about different types of food.
1. Formulating the problem to minimize the cost of food consumption

From the task, we formulize by the problem by decision variables a, b, c, d and e.

\[ a = \text{Bread in 100gr} \]
\[ b = \text{Fish in 100gr} \]
\[ c = \text{Oranges in 100gr} \]
\[ d = \text{Dates in 100gr} \]
\[ e = \text{Butter in 100gr} \]

The objective function is how to minimize the cost for food consumption

\[ 0.12a + 0.35b + 0.06c + 0.62d + 0.52e \rightarrow \text{Min} \]

The Constraints are:

\[ 266a + 96b + 47c + 282d + 717e - 2500 \leq 0 \quad \text{Energy constraint} \]
\[ 3.29a + 1.7b + 0.12c + 0.39d + 81e - 60 \leq 0 \quad \text{Fat constraint} \]
\[ 50.61a + 11.75c + 75.03d + 0.06e - 220 \leq 0 \quad \text{Carbs constraint} \]
\[ 7.64a + 20.08b + 0.94c + 2.45d + 0.85e - 90 \leq 0 \quad \text{Protein constraint} \]
\[ 9c + 0.1d - 9 \leq 0 \quad \text{Vitamin C constraint} \]
2. Solve Mathematical problem using Lp_Solve

The cost of a package is 1,59 euro. It consists of 11,52 x 100 gr of bread, 1 x 100gr orange and 0,266 x 100 gr butter.
3. If the butter is discarded from food package

The cost of a package is 2,244 euro. It consists of 18,20 x 100 gr of bread and 1 x 100 gr orange. The package gets much more expensive as an excessive amount of additional bread is added to satisfy the fat constraint. It might also be unhealthy to eat such a large amount of bread.
4. Formulate problem for serving food package

One serving of food package will cost 1,664 euro that consist of 23 slice of bread, 1 of orange and 4 small package of butter without dates and fishes.
3.1 1. Draw a Hasse diagram for the partially ordered set of points:
\[(\{(1, 2)^T, (2, 1)^T, (2, 2)^T, (5, 4)^T, (4, 4)^T, (5, 3)^T\}, \leq_C)\]
and indicate minimal and maximal elements.
3.2

2. Derive a formula or algorithm for checking analytically whether a point dominates another point.

\[
\begin{align*}
    y'_1 &= y_1 - 1\lambda_1 + 2\lambda_2 \\
    y'_2 &= y_2 + 2\lambda_1 - 1\lambda_2 \\
    \lambda_1 &= y_1 - y'_1 + 2\lambda_2 \\
    \lambda_2 &= (y'_2 - y_2) + 2(y'_1 - y_1) \\
    \lambda_1 &= (y'_1 - y_1) + 2(y'_2 - y_2)
\end{align*}
\]

\[
y \prec_C y', \text{ if and only if } \\
\lambda_2 = (y'_2 - y_2) + 2(y'_1 - y_1) \geq 0 \text{ and } \lambda_1 = (y'_1 - y_1) + 2(y'_2 - y_2) \geq 0.
\]"
3.3 3. Show that \( \preceq_C \) is a partial order.  

(2Points)

Reflexivity: We need to show \((y_1, y_2) \preceq_C (y_1, y_2)\). According to the formula in 3.2 holds if \(y_2 - y_2 + 2(y_1 - y_1) \geq 0\) and \((y_1 - y_1) + 2(y_2 - y_2) \geq 0\). The condition is obviously always satisfied.

Transitivity: We need to show \((y_1^A, y_2^A) \preceq_C (y_1^B, y_2^B)\) and \((y_1^B, y_2^B) \preceq_C (y_1^C, y_2^C)\) implies \((y_1^A, y_2^A) \preceq_C (y_1^C, y_2^C)\). According to the formula in 3.2 this is equivalent to

\[
\begin{align*}
y_2^B - y_2^A + 2(y_1^B - y_1^A) & \geq 0 \\
y_1^B - y_1^A + 2(y_2^B - y_2^A) & \geq 0 \\
y_2^C - y_2^B + 2(y_1^C - y_1^B) & \geq 0 \\
y_1^C - y_1^B + 2(y_2^C - y_2^B) & \geq 0
\end{align*}
\]

implies

\[
\begin{align*}
y_2^B - y_2^A + 2(y_1^B - y_1^A) + y_2^C - y_2^B + 2(y_1^C - y_1^B) & \geq 0 \\
y_1^B - y_1^A + 2(y_2^B - y_2^A) + (y_1^C - y_1^B) + 2(y_2^C - y_2^B) & \geq 0
\end{align*}
\]
Antisymmetry: We need to show \((y_1^A, y_2^A) \preceq_C (y_1^B, y_2^B)\) and 
\((y_1^B, y_2^B) \preceq_C (y_1^A, y_2^A)\) implies \((y_1^A, y_2^A) = (y_1^B, y_2^B)\).
According to the formula in 3.2 this is equivalent to
\[
\begin{align*}
y_2^B - y_2^A + 2(y_1^B - y_1^A) & \geq 0 \\
y_1^B - y_1^A + 2(y_2^B - y_2^A) & \geq 0 \\
y_2^A - y_2^B + 2(y_1^A - y_1^B) & \geq 0 \\
y_1^A - y_1^B + 2(y_2^A - y_2^B) & \geq 0
\end{align*}
\]
\(\Rightarrow y_1^A = y_1^B\) and \(y_1^B = y_1^A\)

We prove by rejecting the contrapositive:
if \(y_1^A > y_1^B\) and \(y_2^A > y_2^B\) then the first and second equation are not satisfied,
if \(y_1^B > y_1^A\) and \(y_2^B > y_2^A\) then the third and fourth equation are not satisfied.
Moreover, if \(y_1^A > y_1^B\) and \(y_2^A < y_2^B\) then
\[
y_2^B - y_1^A + 2(y_1^B - y_1^A) + y_1^A - y_1^B + 2(y_2^A - y_2^B) \geq 0
\]
or 
\[
y_2^B - y_1^A + 2(y_1^B - y_1^A) + y_1^A - y_1^B + 2(y_2^A - y_2^B) \geq 0
\]
or 
\[
y_2^B + y_2^A - y_1^A + y_1^B \geq 0\]
is not satisfied. And, finally if \(y_1^A < y_1^B\) and \(y_2^A > y_2^B\) then
\[
y_1^B - y_1^A + 2(y_2^B - y_2^A) + y_2^A - y_2^B + 2(y_1^A - y_1^B) \geq 0
\]
or 
\[
y_1^B - y_1^A + 2(y_2^B - y_2^A) + y_2^A - y_2^B + 2(y_1^A - y_1^B) \geq 0
\]
or 
\[
y_1^B + y_1^A + y_2^B - y_2^A \geq 0\]
is not satisfied. Hence, the only remaining option is \(y_1^A = y_2^A\) and \(y_1^B = y_2^B\), which is
obviously a feasible solution satisfying all four inequalities.

3. Show that \(\preceq_C\) is a partial order.
4. Is $\preceq_C$ an extension of the weak componentwise order on $\mathbb{R}^2$ or does the weak componentwise order extend $\preceq_C$? Give reasons for your answer.

Yes, because $C$ is a superset of the positive orthant (the cone that gives rise to the weak componentwise order). For if $C$ superset $C'$, then the cone order for $C$ extends the cone order for $C'$. For theorem see slides on cone orders.

5. Find the Pareto front of the multi-objective optimization problem and draw it in a 2-D coordinate system:

$$f_1(x) = x^2, f_2(x) = (1 - x)^2, x \in [0, 1]$$

Hint: Try to express $f_2$ as a function of $f_1$ and analyze the resulting curve.

We derive $f_2$ as a function of $f_1$: $f_2 = (1 - \sqrt{f_1})^2$
6. Find the minimal set of $\preceq_{\mathcal{C}}$ on the Pareto front obtained in the previous subtask.

The main idea is to find the points on the Pareto front where the cone, when added to a point, does not anymore lie above the curve. Then some points outside the cone dominated area are dominating. This happens when the slope of the curve exceeds -2 or is below -1.

$$\frac{\partial f_2}{\partial f_1} = 1 - (f_1)^{-0.5} = 1 - 1/\sqrt{f_1}$$

$$\frac{\partial f_2}{\partial f_1} = 1 - 1/\sqrt{f_1} \geq -2$$
$$\Leftrightarrow f_1 \geq (1/3)^2 = 1/9$$

$$\frac{\partial f_2}{\partial f_1} = 1 - 1/\sqrt{f_1} \cdot -1/2$$
$$\Leftrightarrow f_1 \cdot 4/9$$

7. Discuss the advantages and disadvantages of using $\preceq_{\mathcal{C}}$ as a ordering relation in place of the Pareto dominance relation.

Points with a very unbalanced tradeoff do not enter the minimal set. This is a constraint on the trade-off. However, if there is nothing known about the relative importance of the objective functions, some good points might not be part of the minimal set anymore. In the plot the extreme values would also get lost (range information).