Evolutionary Algorithms
Problem Set - Genetic Algorithms
Solutions

1. (a) The number of possibilities is $\binom{l}{i}$. Thus, for the probability that a randomly generated string contains i ones is:

$$p_i = \binom{l}{i} \cdot p^i \cdot (1-p)^{l-i} \tag{1}$$

(b) With the probability that a randomly generated string contains i ones defined as:

$$p_i = \binom{l}{i} \cdot p^i \cdot (1-p)^{l-i} \tag{2}$$

the expectation $E(X)$ for the number of ones in a randomly generated string is:

$$\begin{aligned}
E(X) &= \sum_{i=0}^{l} i \cdot \binom{l}{i} \cdot p^i \cdot (1-p)^{l-i} \\
&= \sum_{i=1}^{l} \frac{l!}{i! \cdot (l-i)!} \cdot i \cdot p^i \cdot (1-p)^{l-i} \\
&= \sum_{i=1}^{l} \frac{l!}{(i-1)! \cdot (l-i)!} \cdot p^i \cdot (1-p)^{l-i} \\
&= p \cdot l \sum_{i=1}^{l} \frac{(l-1)!}{(i-1)! \cdot (l-i)!} \cdot p^{i-1} \cdot (1-p)^{l-i} \\
&= p \cdot l \sum_{i=0}^{l-1} \frac{(l-1)!}{i! \cdot (l-i-1)!} \cdot p^i \cdot (1-p)^{l-i-1} \\
&= p \cdot l \sum_{i=0}^{l-1} \binom{l-1}{i} \cdot p^i \cdot (1-p)^{l-i-1} \\
&= p \cdot l \cdot (p + (1-p))^{l-1} \\
&= p \cdot l \cdot 1 \\
&= p \cdot l
\end{aligned}$$

2. (a) For the string (0101) and mutation operator p, the probabilities for the 16 possible mutations that can happen are given by:

$$\begin{aligned}
p(0101 \rightarrow 0000) &= (1-p)^2 \cdot p^2 \\
p(0101 \rightarrow 0001) &= (1-p)^3 \cdot p \\
p(0101 \rightarrow 0010) &= (1-p) \cdot p^3 \\
p(0101 \rightarrow 0011) &= (1-p)^2 \cdot p^2 \\
p(0101 \rightarrow 0100) &= (1-p)^3 \cdot p \\
p(0101 \rightarrow 0101) &= (1-p)^4 \\
p(0101 \rightarrow 0110) &= (1-p)^2 \cdot p^2 \\
p(0101 \rightarrow 0111) &= (1-p)^3 \\
p(0101 \rightarrow 1000) &= (1-p) \cdot p^3
\end{aligned}$$

$$
\begin{aligned}
p(0101 \to 1001) &= (1-p)^2 \cdot p^2 \\
p(0101 \to 1010) &= p^4 \\
p(0101 \to 1011) &= (1-p) \cdot p^3 \\
p(0101 \to 1100) &= (1-p)^2 \cdot p^2 \\
p(0101 \to 1101) &= (1-p)^3 \cdot p \\
p(0101 \to 1110) &= (1-p) \cdot p^3 \\
p(0101 \to 1111) &= (1-p)^2 \cdot p^2
\end{aligned}
$$

With this, the probability that 2 out of the 4 bits are changed ($P(X = 2)$) is the sum of the possibilities in which 2 bits have changed. From the equations above, we can see that there are 6 of such possibilities, thus we get:

$$P(X = 2) = 6 \cdot (1-p)^2 \cdot p^2 \tag{3}$$

(b) In the more general case for strings of length l and a change of i bits, the probability that i out of the l bits are changed ($P(X = i)$) is:

$$P(X = i) = \binom{l}{i} \cdot (1-p)^{l-i} \cdot p^i \tag{4}$$

3. (a) Schema $H = (0 * * 1 * 1 * * 0 * * *)$ has order $o(H) = 4$ and defining length $d(H) = 8$.

(b) The probability p_{s_c} that schema H survives one-point crossover is defined as:

$$p_{s_c} = 1 - p_c \frac{d(H)}{l-1} \tag{5}$$

With $p_c = 0.3$, $d(H) = 8$ and $l = 12$ we get:

$$p_{s_m} = 1 - 0.3 \cdot \frac{8}{11} \approx 0.78 \tag{6}$$

(c) The probability p_{s_m} that schema H survives mutation is defined as:

$$p_{s_m} = (1 - p_m)^{0(H)} \tag{7}$$

With $p_m = 1/12$ and $o(H) = 4$ we get:

$$p_{s_m} = (1 - 1/12)^4 \approx 0.71 \tag{8}$$