Logica (I&E)

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`http://liacs.leidenuniv.nl/~vlietrvan1/logica/`

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1.4 Semantics of propositional logic
1.2 Natural deduction

Ik hou van werken zolang het werken is waarvan ik hou.

A slide from lecture 1:

1.4. Semantics of propositional logic

Definition 1.28.

1. The set of truth values contains two elements T and F, where T represents ‘true’ and F represents ‘false’.
2. A *valuation* of *model* of a formula ϕ is an assignment of each propositional atom in ϕ to a truth value.

Part of a slide from lecture 1:

(4) All Martians like pepperoni on their pizza.

A slide from lecture 1:

Truth table for conjunction

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

Truth tables

ϕ	ψ	$\phi \wedge \psi$
T	T	T
T	F	F
F	T	F
F	F	F

ϕ	ψ	$\phi \vee \psi$
T	T	T
T	F	T
F	T	T
F	F	F

ϕ	ψ	$\phi \rightarrow \psi$
$\ldots$	$\ldots$	$\ldots$

ϕ	$\neg\phi$
T	F
F	T

Truth table for implication

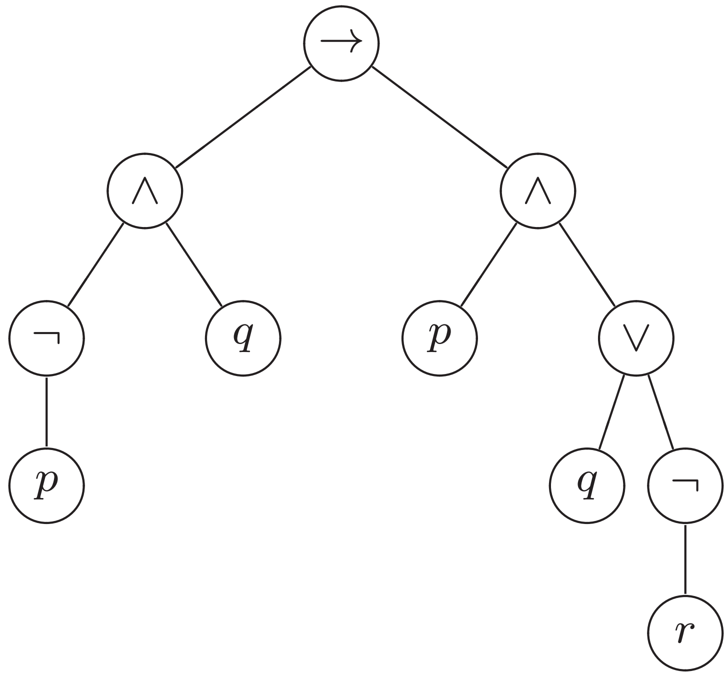
ϕ	ψ	$\phi \to \psi$
T	T	T
T	F	F
F	T	T
F	F	T

ϕ	ψ	$\neg\phi \vee \psi$
T	T	T
T	F	F
F	T	T
F	F	T

Semantically equivalent

Determining truth value in tree

$\neg p \wedge q \rightarrow p \wedge (q \vee \neg r)$



$n = 3$, so 2^3 lines in truth table

p : T $\qquad$ q : F $\qquad$ r : T

Determining truth value in table

$(p \to \neg q) \to (q \vee \neg p)$

p	q	$\neg p$	$\neg q$	$p \to \neg q$	$q \vee \neg p$	$(p \to \neg q) \to (q \vee \neg p)$
T	T	...	...	...	...	...
T	F	...	...	...	...	...
F	T	...	...	...	...	...
F	F	...	...	...	...	...

Determining truth value in table

$(p \to \neg q) \to (q \vee \neg p)$

p	q	$\neg p$	$\neg q$	$p \to \neg q$	$q \vee \neg p$	$(p \to \neg q) \to (q \vee \neg p)$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

1.4.3. Soundness of propositional logic

Definition 1.34.
If, for all valuations in which all $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to T, ψ evaluates to T as well, we say that

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds and call $\vDash$ the *semantic entailment* relation.

Examples semantic entailment

1. $p \wedge q \models p$?

2. $p \vee q \models p$?

3. $\neg q, p \vee q \models p$?

4. $p \models q \vee \neg q$?

1.4.2. Mathematical induction

$$1 + 2 + 3 + 4 + \cdots + n = \ldots$$

Mathematical induction

For property M of natural numbers:

1. Base case: The natural number 1 has property M, i.e., we have a proof of $M(1)$

2. Inductive step: If n is a natural number which *we assume* to have property $M(n)$, then *we can show* that $n+1$ has property $M(n+1)$; i.e., we have a proof of $M(n) \rightarrow M(n+1)$.

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Definition 1.30. The principle of mathematical induction says that, on the grounds of these two pieces of information above, every natural number n has property $M(n)$.
The assumption of $M(n)$ in the inductive step is called the *induction hypothesis*.

Natural numbers

Mathematics: $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$

Computer science: $\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}$

Theorem 1.31. The sum $1+2+3+4+\cdots+n$ equals $n\cdot(n+1)/2$ for all natural numbers n.

Proof: $\mathsf{LHS}_n = \mathsf{RHS}_n \ldots$

Definition. Let the level of the root in a binary tree be 1, the level of the children of the root be 2, . . . (N.B.: different from Algoritmiek). The *height* of a binary tree is the maximum level of the tree. A binary tree of height h is called *filled*, if every level of the tree contains the maximum number of nodes.

Exercise. Prove by induction that
(a) for each level l of a filled binary tree, the number of nodes at level l equals 2^{l-1},
(b) the number of nodes in a filled binary tree of height h equals $2^h - 1$,
(c) the maximum number of swaps needed for (bottom-up) heapify in a filled binary tree of height h equals $2^h - 1 - h$.

Variants of induction

Mathematical induction:

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2. Inductive step: If n is a natural number which *we assume* to have property $M(n)$, then *we can show* that $n+1$ has property $M(n+1)$; i.e., we have a proof of $M(n) \rightarrow M(n+1)$.

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Course-of-values induction:

2. Inductive step: If n is a nonnegative, integer number for which *we assume* that $M(1) \land M(2) \land \cdots \land M(n)$ holds, then *we can show* that $n+1$ has property $M(n+1)$; i.e., we have a proof of $M(1) \land M(2) \land \cdots \land M(n) \to M(n+1)$.

Fibonacci

(variant of Exercise 1.4.8)

$F_1 = 1$,
$F_2 = 1$,
$F_{n+1} = F_n + F_{n-1}$ if $n \geq 2$

Use course-of-values induction to prove that F_n is even, if and only if $n \equiv 0 \pmod 3$.

Variants of induction

Course-of-values induction:

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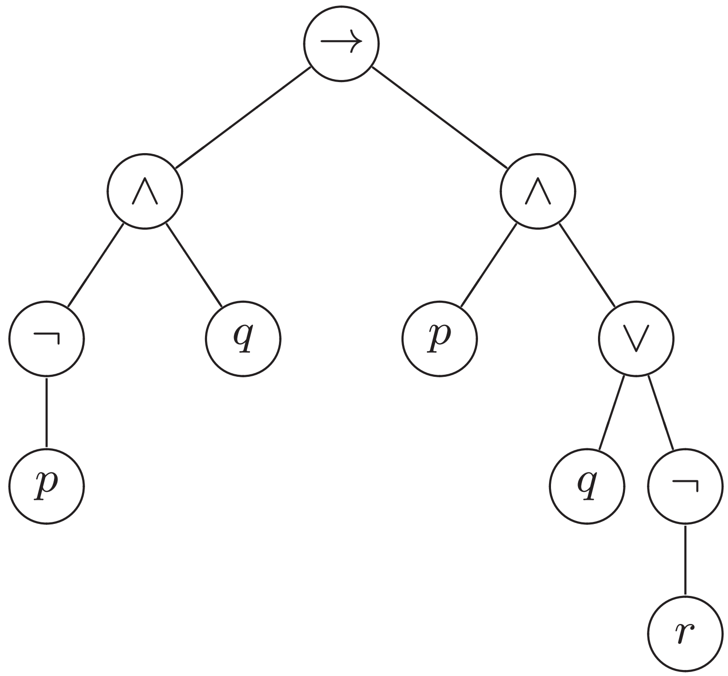
Structural induction: induction on the structure

Formulas, trees, . . .

$$(((\neg p) \land q) \to (p \land (q \lor (\neg r))))$$



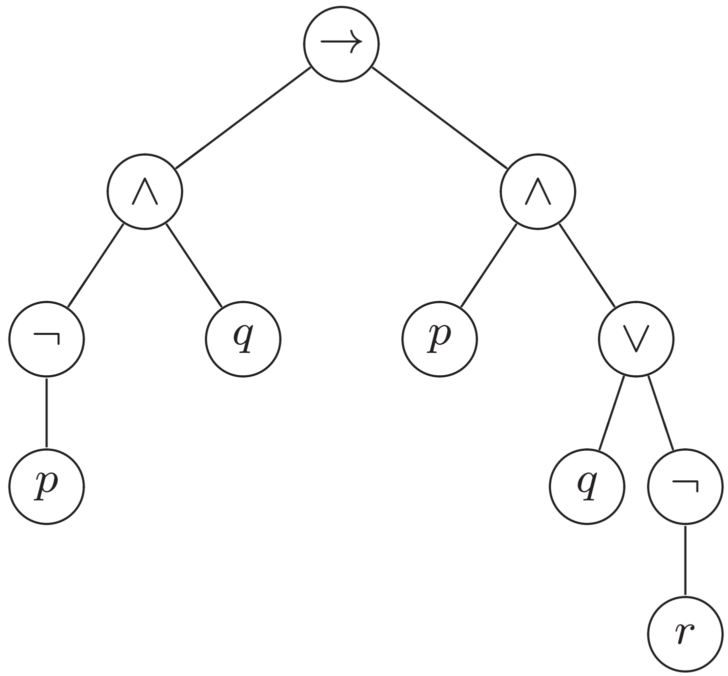
Definition 1.32. Given a well-formed formula ϕ, we define its height to be 1 plus the length of the longest path of its parse tree.

Brackets in a well-formed formula

Theorem 1.33.
For every well-formed propositional logic formula, the number of left brackets is equal to the number of right brackets.

Proof. . .

$$(((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r))))$$



Mathematical induction would not work. . .

1.2. Natural deduction

Proof rules

Premises $\phi_1, \phi_2, \ldots, \phi_n$

Conclusion ψ

Sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

The rules for conjunction

And-introduction:

$$\frac{\phi \qquad \psi}{\phi \wedge \psi} \wedge\mathsf{i}$$

The rules for conjunction

And-elimination:

$$\frac{\phi \land \psi}{\phi} \land e_1 \qquad\qquad \frac{\phi \land \psi}{\psi} \land e_2$$

Example 1.4. Proof of: $p \wedge q, r \ \vdash \ q \wedge r$

Example 1.4. Proof of: $p \wedge q, r \vdash q \wedge r$

1	$p \wedge q$	premise
2	r	premise
3	q	$\wedge$e$_2$ 1
4	$q \wedge r$	$\wedge$i 3, 2

In tree-like form...