LIACS — Examination Logic

Leiden University — 10 July 2015, 10:00 - 13:00

1. **[1 point]** Prove by induction that $\sum_{k=1}^{n}(2k-1) = n^2$ for all positive integers n ≥ 1.
   Base case: for n = 1 the left hand side of the equation is 1 and the right hand side is $1^2 = 1$.
   Induction step: Let i ≥ 1 and assume that for n =i the above equation holds.We have
   $$\sum_{k=1}^{i+1}(2k-1) = \sum_{k=1}^{i}(2k-1) + (2(i+1)-1) = i^2 + 2i + 1 = (i+1)^2.$$
   Here we have used the induction hypothesis in one but the last equality.

2. **[2 points]** Give a proof in natural deduction for each of the following sequents:
   a) $\neg p \wedge (q \vee r) \vdash q \to (r \to \neg p)$

1	$\neg p \wedge (q \vee r)$	premise
2	q	assumption
3	r	assumption
4	$\neg p$	$\wedge$eL 3,2
5	$r \to \neg p$	$\to$i 3-4
6	$q \to (r \to \neg p)$	$\to$i 2-5

   b) $p \to q, \neg(q \vee r) \vdash \neg p$

1	$p \to q$	premise
2	$\neg(q \vee r)$	premise
3	p	assumption
4	q	$\to e_L$ 3,1
5	$q \vee r$	$\vee$i 4
6	$\perp$	$\neg$e 2,5
7	$\neg p$	$\neg$i 2-5

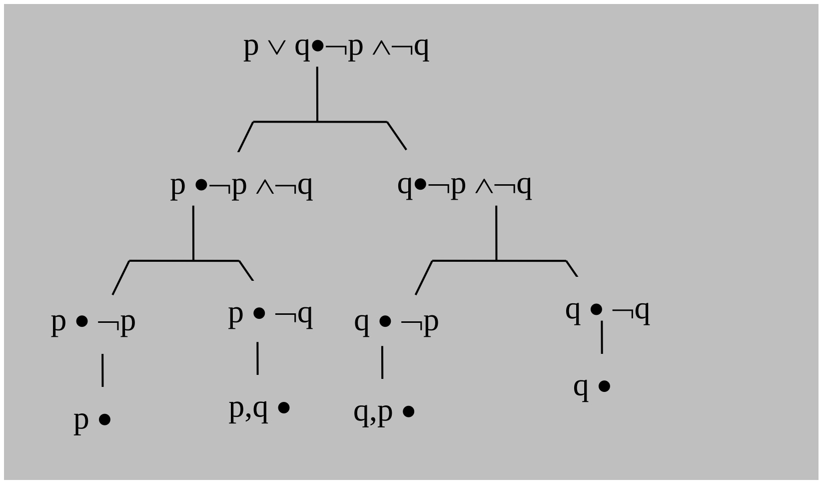
   c) $p \wedge q, \neg p \vdash \neg q \to p$

1	$p \wedge q$	premise
2	$\neg p$	premise
3	p	$\wedge e_L$ 1
4	$\perp$	$\neg$e 2,3
5	$\neg q \to p$	$\perp$e 4

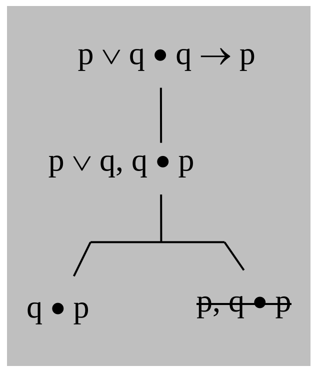
   d) $p \to (p \to (p \to q)) \vdash \neg q \to \neg p$

1	$p \to (p \to (p \to q))$	premise
2	$\neg q$	assumption
3	p	assumption
4	$p \to (p \to q)$	$\to$e 3,1
5	$p \to q$	$\to$e 3,4
6	q	$\to$e 3,5
7	$\perp$	$\neg$e 2,6
8	$\neg p$	$\neg$i 3-7
9	$\neg q \to \neg p$	$\to$i 1-8

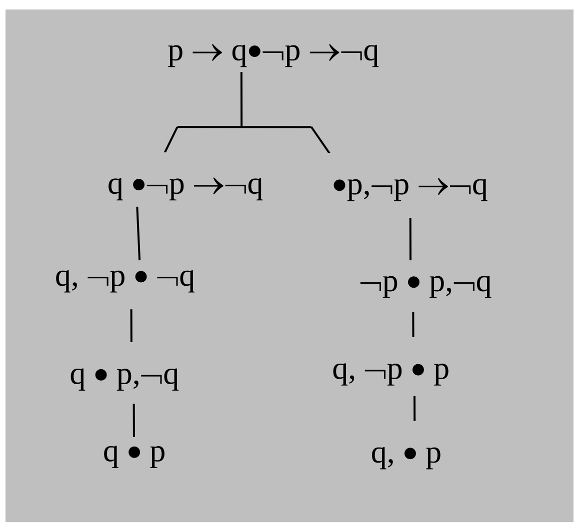
3. **[1,5 points]** Give a semantic tableau to show that the following sequents are not valid:
   a) $p \vee q \vdash \neg p \wedge \neg q$



b) $p \vee q \vdash q \rightarrow p$



c) $p \rightarrow q \vdash \neg p \rightarrow \neg q$



4. **[1 point]** Consider the following truth table for the formulas ϕ and ψ:

p	q	ϕ	ψ
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	F

Find propositional logic formulas in *conjunctive normal form* equivalent to ϕ and ψ, respectively.

The equivalent formula in CNF for ϕ can be obtained as conjunction of the clauses stemming from lines 1 and 2: $(\neg p \vee \neg q) \wedge (\neg p \vee q)$. A formula in CNF for ψ is obtained as conjunction of the clauses stemming from lines 1, 3 and 4: $(\neg p \vee \neg q) \wedge (p \vee \neg q) \wedge (p \vee q)$.

5. **[1,5 points]**
   a) Give a predicate logic formula ϕ expressing the fact that there are at least two elements.
   The formula $\phi \equiv \exists x \exists y \neg(x=y)$ holds in all models with at least two elements.
   b) Give a predicate logic formula ϕ expressing the fact that there are exactly two elements.
   The formula $\phi \equiv \exists x \exists y(\neg(x=y) \wedge \forall z(x=z \vee y=z))$ holds in all models with exactly two elements.
   c) Give a predicate logic formula ϕ such that $\phi[y/x]$ is not the same as $(\phi[z/x])[y/z]$.
   Take any formula where there are free occurrences of both x and z. For example, consider the formula $\phi \equiv x=z$. Then $\phi[y/x] \equiv y=z$ whereas $(\phi[z/x])[y/z] \equiv (z=z)[y/z] \equiv (y = y)$.

6. **[1 point]** Write a formula ϕ in predicate logic such that, for each of the following pair of models M and N, ϕ holds in the model M but not in the model N.
   a) $M = (\mathbb{Q}, P^M)$ and $N = (\mathbb{Z}, ,P^N)$. Here $\mathbb{Q}$ is the set of rational numbers, $\mathbb{Z}$ is the set of integers, and P^M is the strict (thus not equal) order relation < between rational numbers, and, P^N is the strict order relation < between integer number.
   Consider the formula $\forall x \forall y(P(x,y) \rightarrow \exists z\, (P(x,z) \wedge P(z,y))$. It holds in M because for every pairs of rational numbers p and q with p strictly smaller than q, we can take the number r=(p+q):p. The number r is always strictly greater than p, but strictly smaller than q. However the formula does not hold in N as there is, for example, no integer strictly in between 3 and 4.
   b) $M = (\mathbb{Z}, P^M)$ and $N = (\mathbb{Z}, P^N)$. Here $\mathbb{Z}$ is the set of integers, and P^M is the strict (thus not equal) order relation < between integers, and, P^N is the less or equal order relation $\leq$ between integers.
   Consider the formula $\forall x \forall y(Px,y) \rightarrow \neg(x=y))$. It holds in M because for every pairs of integers n and m, if n is strictly smaller than m then n is not equal to m. Clearly this is not true if n is less or equal to m, and the formula does not hold in the model N.

7. **[2 points]** Show the validity of each of the following sequent by means of a proof in natural deduction, where P, Q, are predicates of arity 1, and R is a predicate of arity 2:
   a) $\forall y \neg P(y), (Q(y) \vee R(y,y)) \rightarrow P(x) \vdash \exists x \neg(Q(x) \vee R(x,x))$

1	$\forall y \neg P(y)$	premise
2	$(Q(y) \vee R(y,y)) \rightarrow P(x)$	premise
3	$\neg P(x)$	$\forall$e 1
4	$\neg(Q(y) \vee R(y,y))$	MT, 2,4
5	$\exists x \neg(Q(x) \vee R(x,x))$	$\exists$i 4

   b) $\exists x \forall y R(x,y) \vdash \forall y \exists x R(x,y)$

1			$\exists x \forall y R(x,y)$	premise
2	y_0			
3		x_0	$\forall y R(x_0,y)$	assumption
4			$R(x_0,y_0)$	$\forall$e 3
5			$\exists x R(x,y_0)$	$\exists$i 4
6			$\exists x R(x,y_0)$	$\exists$e 1,3-5
7			$\forall y \exists x R(x,y)$	$\forall$i 2-6

c) $P(x) \rightarrow \forall y Q(y) \vdash \forall y(P(x) \rightarrow Q(y))$

1		$P(x) \rightarrow \forall y Q(y)$	premise
2	y_0		
3		$P(x)$	assumption
4		$\forall y Q(y)$	$\rightarrow$e 1,3
5		$Q(y_0)$	$\forall$e 4
6		$P(x) \rightarrow Q(y_0)$	$\rightarrow$i 3-5
7		$\forall y(P(x) \rightarrow Q(y))$	$\exists$e 1, 3-10

The final score is given by the sum of the points obtained.