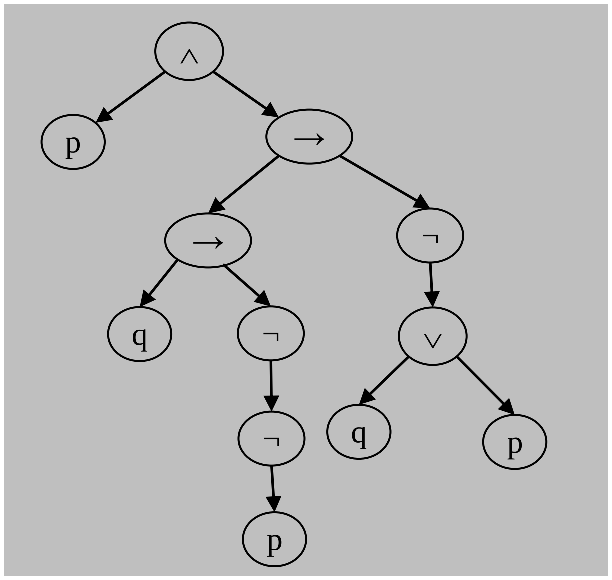
LIACS — Examination Logic

Leiden University — 15 June 2015, 14:00 - 17:00

1. **[1 point]** Draw the parse tree of the formula p ∧ ((q → ¬¬p) → ¬(q ∨ p)) and list *all* its sub-formulas.



The sub-formulas are:
p ∧ ((q → ¬¬p) → ¬(q ∨ p))
p
(q → ¬¬p) → ¬(q ∨ p)
(q → ¬¬p)
¬(q ∨ p)
q
¬¬p
¬p
q ∨ p.

2. **[2 points]** Give a proof in natural deduction for each of the following sequents:

a) ¬p ∨ q, ¬p → q ⊢ q

1	¬p ∨ q	premise		
2	¬p → q	premise		
3	¬p	assumption	q	assumption
4	q	→e 3,2		
5	q	∨e 1,3-4,3		

b) p →(¬p ∧ q) ⊢ ¬p

1	p →(¬p ∧ q)	premise
2	p	assumption
3	¬p ∧ q	→e 2,1
4	¬p	$\wedge e_L$ 3
5	⊥	¬e 2,4
6	¬p	¬i 2-5

c) p ∧ q, ¬(p ∧ r) ⊢ p ∧ ¬r

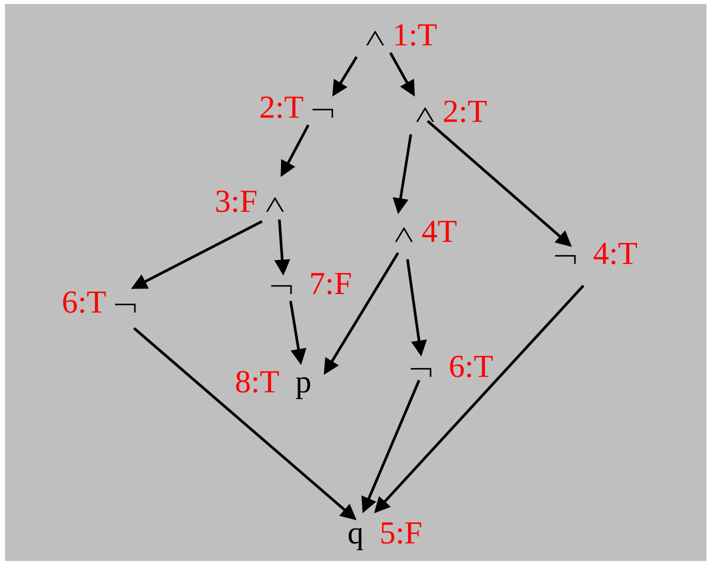
1	p ∧ q	premise
2	¬(p ∧ r)	premise
3	p	$\wedge e_L$ 3
4	r	assumption
5	p ∧ r	∧i 3,4
6	⊥	¬e 2,5
7	¬r	¬i 4-6
8	p ∧ ¬r	∧i 3-7

d) ⊢ (p → q) → ((r → p) → (r → q))

1	$p \to q$	assumption
2	$r \to p$	assumption
3	r	assumption
4	p	$\to$e 3,2
5	q	$r \to q$
6	$r \to q$	$\to$i 3-5
7	$(r \to p) \to (r \to q)$	$\to$i 2-6
8	$(p \to q) \to ((r \to p) \to (r \to q))$	$\to$i 1-7

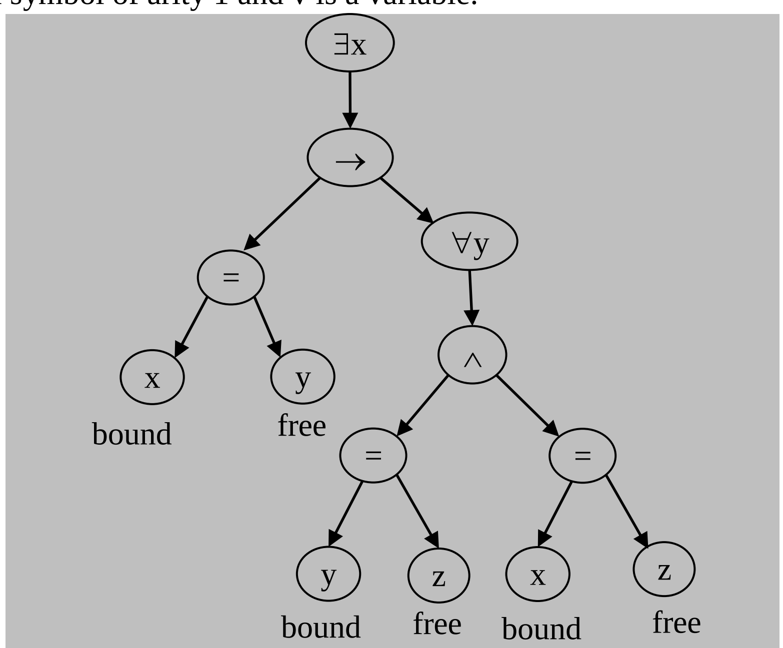
3. **[1,5 points]** Apply the marking algorithm to find a valuation witness for the satisfiability of the following Horn formulas:
   a) $(\top \to p) \land (p \land q \to r) \land (p \to q) \land (q \land r \to s) \land (r \to \bot)$
      1st round: $(\top \to p) \land (p \land q \to r) \land (p \to q) \land (q \land r \to s) \land (r \to \bot)$
      2nd round: $(\top \to p) \land (p \land q \to r) \land (p \to q) \land (q \land r \to s) \land (r \to \bot)$
      3rd round: $(\top \to p) \land (p \land q \to r) \land (p \to q) \land (q \land r \to s) \land (r \to \bot)$
      4th round: $(\top \to p) \land (p \land q \to r) \land (p \to q) \land (q \land r \to s) \land (r \to \bot)$
      5th round: $(\top \to p) \land (p \land q \to r) \land (p \to q) \land (q \land r \to s) \land (r \to \bot)$
      Since $\bot$ is marked, the formula is not satisfiable.
   b) $(p \land q \to r) \land (q \to p) \land (r \land p \to q) \land (r \to s) \land (\top \to p)$
      1st round: $(p \land q \to r) \land (q \to p) \land (r \land p \to q) \land (r \to s) \land (\top \to p)$
      2nd round: $(p \land q \to r) \land (q \to p) \land (r \land p \to q) \land (r \to s) \land (\top \to p)$
      Nothing else can be marked, so the formula is satisfiable with a valuation mapping p to true and all other atomic propositions to false.
   c) $(p \land q \land r \to s) \land (p \land q \to r) \land (r \to q) \land (p \to \bot) \land (\top \to r)$
      1st round: $(p \land q \land r \to s) \land (p \land q \to r) \land (r \to q) \land (p \to \bot) \land (\top \to r)$
      2nd round: $(p \land q \land r \to s) \land (p \land q \to r) \land (r \to q) \land (p \to \bot) \land (\top \to r)$
      3rd round: : $(p \land q \land r \to s) \land (p \land q \to r) \land (r \to q) \land (p \to \bot) \land (\top \to r)$
      Nothing else can be marked, so the formula is satisfiable with a valuation mapping q and r to true and all other atomic propositions to false.

4. **[1 point]** Draw the DAG corresponding to the formula $\neg(\neg q \land \neg p) \land ((p \land \neg q) \land \neg q)$ and use a SAT solver to give a witness for its satisfiability.
   A DAG for this formula (and a valuation for its satisfiability) is



5. **[1,5 points]** Let ϕ be the formula $\exists x(x=y \rightarrow \forall y(y=z \wedge x=z))$ where x,y,z are three variables. Draw the parse tree of ϕ and compute, when possible, the following substitutions:
   - $\phi[f(v)/y]$
   - $\phi[f(y)/y]$
   - $\phi[f(v)/z]$.

   Here f is a function symbol of arity 1 and v is a variable.



- $\phi[f(v)/y] = \exists x(x=f(v) \rightarrow \forall y(y=z \wedge x=z))$
- $\phi[f(y)/y] = \exists x(x=f(y) \rightarrow \forall y(y=z \wedge x=z))$
- $\phi[f(v)/z] = \exists x(x=y \rightarrow \forall y(y=f(v) \wedge x=f(v)))$

6. **[1 points]** Find a model for each of the following sequent showing that it is not valid.
   a) $\exists x P(x), \exists x Q(x) \vdash \exists x(P(x) \wedge Q(x))$, where P and Q are predicates of arity 1.

Consider the model M where $A = \{a,b\}$, $P^M=\{a\}$, and $Q^M=\{b\}$. Then the two leftmost formulas are both true but the rightmost one is not.

b) $\forall x \forall y(\neg x=y \to (P(x) \land P(y))) \vdash \forall x P(x)$, where P is a predicate of arity 1.

Consider the model M where $A = \{a\}$, $P^M=\emptyset$. Then the leftmost formula is true (because $\neg x=y$ is false for all elements of the universe) but the rightmost one is not.

7. **[2 points]** Show the validity of each of the following sequent by means of a proof in natural deduction, where P, Q, are predicates of arity 1, and R is a predicate of arity 2:

a) $\forall x \forall y(x=y \to R(x,y)) \vdash \forall x R(x,x)$,

1		$\forall x \forall y(x=y \to R(x,y))$	premise
2	x_0	$\forall y(x_0=y \to R(x_0,y))$	$\forall$e 1
3		$x_0=x_0 \to R(x_0,x_0)$	$\forall$e 1
4		$x_0=x_0$	=i
5		$R(x_0, x_0)$	$\to$e 4,3
6		$\forall x R(x,x)$	$\forall$i 1-5

b) $\exists x(P(x) \land \neg Q(x)), \exists x(\neg P(x) \land Q(x)) \vdash \exists x \exists y(P(x) \land Q(y))$

1			$\exists x(P(x) \land \neg Q(x))$	premise
2			$\exists x(\neg P(x) \land Q(x))$	premise
3	x_0		$P(x_0) \land \neg Q(x_0)$	assumption
4		y_0	$\neg P(y_0) \land Q(y_0)$	assumption
5			$P(x_0)$	$\land e_R$ 3
6			$Q(y_0)$	$\land e_L$ 4
7			$P(x_0) \land Q(y_0)$	$\land$i 5, 6
8			$\exists y(P(x_0) \land Q(y))$	$\exists$i 7
9			$\exists y(P(x_0) \land Q(y))$	$\exists$e 2, 4-8
10			$\exists x \exists y(P(x) \land Q(y))$	$\exists$i 9
11			$\exists x \exists y(P(x) \land Q(y))$	$\exists$e 1, 3-10

The final score is given by the sum of the points obtained.