Logica (I&E)

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1.4 Semantics of propositional logic

Semantische tableaus

Hoe simpeler hoe beter. Want hoe minder keuze je een speler laat, hoe groter de kans dat hij het juiste doet.

Part of a slide from lecture 5:

Theorem 1.35. (Soundness)

Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional logic formulas.
If

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

is valid, then

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds.

Step 2:

If

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

is valid, then

Step 1: $\quad \vDash \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots (\phi_n \to \psi) \ldots)))$

Step 2: $\quad \vdash \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots (\phi_n \to \psi) \ldots)))$

Step 3: $\quad \phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

Theorem 1.37.

If $\vDash \eta$ holds, then $\vdash \eta$ is valid.

In other words, if η is a tautology, then η is a theorem.

A slide from lecture 5:

'Encode' each line in the truth table of η as a sequent.

Proposition 1.38.
Let ϕ be a formula such that $p_1, p_2, \ldots, p_m$ are its only propositional atoms.
Let l be any line in ϕ's truth table.
For all $1 \leq i \leq m$, let $\widehat{p}_i$ be p_i if the entry in line l of p_i is T, otherwise $\widehat{p}_i$ is $\neg p_i$.
Then we have

1. $\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m \vdash \phi$ is provable if the entry for ϕ in line l is T

2. $\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m \vdash \neg\phi$ is provable if the entry for ϕ in line l is F

Proof: by structural induction on formula $\phi \ldots$

A slide from lecture 5:

Example.

$m = 7$

p_1	p_2	p_3	p_4	p_5	p_6	p_7	ϕ	provable sequent
T	T	T	T	T	T	T	T	$p_1, p_2, p_3, p_4, p_5, p_6, p_7 \vdash \phi$
$\color{red}T$	$\color{red}T$	$\color{red}F$	$\color{red}T$	$\color{red}F$	$\color{red}F$	$\color{red}T$	T	$\color{red}p_1, p_2, \neg p_3, p_4, \neg p_5, \neg p_6, p_7 \vdash \phi$
T	F	F	F	T	T	F	T	$p_1, \neg p_2, \neg p_3, \neg p_4, p_5, p_6, \neg p_7 \vdash \phi$
F	F	F	F	F	F	F	T	$\neg p_1, \neg p_2, \neg p_3, \neg p_4, \neg p_5, \neg p_6, \neg p_7 \vdash \phi$
...	...	...	...	...	...	...	...	...
T	T	T	F	T	F	F	F	$p_1, p_2, p_3, \neg p_4, p_5, \neg p_6, \neg p_7 \vdash \neg\phi$
F	T	T	F	T	T	T	F	$\neg p_1, p_2, p_3, \neg p_4, p_5, p_6, p_7 \vdash \neg\phi$

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Step 2:

If

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi$$

is valid, then

Step 1: $\quad \models \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots (\phi_n \to \psi) \ldots)))$

Step 2: $\quad \vdash \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots (\phi_n \to \psi) \ldots)))$

Step 3: $\quad \phi_1, \phi_2, \ldots, \phi_n \vdash \psi$

Theorem 1.37.

If $\models \eta$ holds, then $\vdash \eta$ is valid.

In other words, if η is a tautology, then η is a theorem.

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Let ϕ be a formula such that $p_1, p_2, \ldots, p_m$ are its only propositional atoms.
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For all $1 \le i \le m$, let $\widehat{p}_i$ be p_i if the entry in line l of p_i is T, otherwise $\widehat{p}_i$ is $\neg p_i$.
Then we have

1. $\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m \vdash \phi$ is provable if the entry for ϕ in line l is T

2. $\widehat{p}_1, \widehat{p}_2, \ldots, \widehat{p}_m \vdash \neg\phi$ is provable if the entry for ϕ in line l is F

Consider

$$\phi = \phi_1 \to (\phi_2 \to (\phi_3 \to (\ldots (\phi_n \to \psi) \ldots)))$$

Example.

$$p \wedge q \rightarrow p$$

Corollary 1.39. (Soundness and Completeness)

Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be formulas of propositional logic.
Then

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds, iff the sequent

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

is valid.

3.2. Semantische Tableaus

From [Van Benthem et al., 2003]:

Gevolgtrekking:

$$\phi_1, \ldots, \phi_n \ / \ \psi$$

'ψ volgt logisch uit $\phi_1, \phi_2, \ldots, \phi_n$'

Model:
Definitie 2.5. Een waardering V heet een *model* van een formule ϕ als geldt: $V(\phi) = 1$. (1 = T, 0 = F)

Model van formuleverzameling:
Definitie 2.6. Een waardering V heet een *model* van een formuleverzameling $\Sigma = \{\phi_1, \ldots, \phi_n\}$ als $V(\phi_i) = 1$ voor elke $\phi_i \in \Sigma$.

3.2. Semantische Tableaus

From [Van Benthem et al., 2003]:

Sequent:

$$\phi_1, \ldots, \phi_n \circ \psi_1, \ldots, \psi m$$

Met $m, n \geq 0$

Tegenvoorbeeld:

Een waardering V heet een *tegenvoorbeeld* van een sequent

$$\phi_1, \ldots, \phi_n \circ \psi_1, \ldots, \psi_m$$

als $V(\phi_1) = \ldots = V(\phi_n) = 1$ en $V(\psi_1) = \ldots = V(\psi_m) = 0$.

Voorbeeld 3.4.

$$\neg q, p \rightarrow q \;\;/\;\; \neg p$$

Voorbeeld 3.4.

$$\neg p, p \rightarrow q \ / \ \neg q$$

Voorbeeld 3.5.

$$p \lor q, p \to r, q \to s \; / \; r \lor s$$

Full notation

Voorbeeld 3.6.

$$p \vee q, p \rightarrow r, q \rightarrow s \ / \ r \vee s$$

Simplified notation

Voorbeeld 3.7.

$$(p \rightarrow q) \rightarrow (p \rightarrow r) \ / \ q \rightarrow r$$

Reductieregels

$\neg_L$:
$$\Phi, \neg\alpha \circ \Psi$$
$$\Phi \circ \alpha, \Psi$$

$\neg_R$:
$$\Phi \circ \neg\alpha, \Psi$$
$$\Phi, \alpha \circ \Psi$$

$\wedge_L$:
$$\Phi, \alpha \wedge \beta \circ \Psi$$
$$\Phi, \alpha, \beta \circ \Psi$$

$\wedge_R$:
$$\Phi \circ \alpha \wedge \beta, \Psi$$
$$\Phi \circ \alpha, \Psi \qquad \Phi \circ \beta, \Psi$$

$\vee_L$:
$$\Phi, \alpha \vee \beta \circ \Psi$$
$$\Phi, \alpha \circ \Psi \qquad \Phi, \beta \circ \Psi$$

$\vee_R$:
$$\Phi \circ \alpha \vee \beta, \Psi$$
$$\Phi \circ \alpha, \beta, \Psi$$

$\rightarrow_L$:
$$\Phi, \alpha \rightarrow \beta \circ \Psi$$
$$\Phi, \beta \circ \Psi \qquad \Phi \circ \alpha, \Psi$$

$\rightarrow_R$:
$$\Phi \circ \alpha \rightarrow \beta, \Psi$$
$$\Phi, \alpha \circ \beta, \Psi$$

Open tableau – gesloten tableau

Voorbeeld 3.8.

$$p \rightarrow (q \rightarrow r) \;/\; (p \rightarrow q) \rightarrow r$$