Logica (I&E)

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http://liacs.leidenuniv.nl/~vlietrvan1/logica/

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2. Predicate logic
2.4. Semantics of predicate logic

We zijn op zoek gegaan naar de overwinning en dan kom je hem vanzelf tegen.

A slide from lecture 10:

Definition 2.8.
Given a term t, a variable x and a formula ϕ,
we say that t is free for x in ϕ,
if no free x leaf in ϕ occurs in the scope of $\forall y$ or $\exists y$ for any variable y occurring in t.

If no free occurrences of x in ϕ...

If t is not free for x in ϕ...

2.4. Semantics of predicate logic

In propositional logic:

A slide from lecture 6:

Corollary 1.39. (Soundness and Completeness)
Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be formulas of propositional logic. Then

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds, iff the sequent

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

is valid.

Truth values for

$$(p \vee \neg q) \to (q \to p)$$

Truth values for

$$\forall x \exists y ((P(x) \vee \neg Q(y)) \to (Q(x) \to P(y)))$$

?

Or for

$$P(t_1, t_2, \ldots, t_n)$$

?

Definition 2.14.
Let $\mathcal{F}$ be a set of function symbols and $\mathcal{P}$ a set of predicate symbols, each symbol with a fixed arity.
A model of the pair $(\mathcal{F}, \mathcal{P})$ consists of the following set of data:

1. A non-empty set A, the universe of concrete values;
2. for each nullary symbol $f \in \mathcal{F}$, a concrete element $f^{\mathcal{M}}$ of A;
3. for each $f \in \mathcal{F}$ with arity $n > 0$, a concrete function $f^{\mathcal{M}} : A^n \to A$ from A^n, the set of n-tuples over A, to A;
4. for each $P \in \mathcal{P}$ with arity $n > 0$, a subset $P^{\mathcal{M}} \subseteq A^n$ of n-tuples over A;
5. $=^{\mathcal{M}}$ is equality on A

Example 2.15.
$\mathcal{F} \stackrel{\text{def}}{=} \{i\}$ (nullary)
$\mathcal{P} \stackrel{\text{def}}{=} \{R, F\}$ (binary, unary)

Model $\mathcal{M}$:
$A \stackrel{\text{def}}{=} \{a, b, c\}$ (states in computer program)
$i^{\mathcal{M}} \stackrel{\text{def}}{=} a$, $R^{\mathcal{M}} \stackrel{\text{def}}{=} \{(a,a),(a,b),(a,c),(b,c),(c,c)\}$ $F^{\mathcal{M}} \stackrel{\text{def}}{=} \{b, c\}$

1. Informal model check of formula

$$\exists y R(i, y)$$

Example 2.15.
$\mathcal{F} \stackrel{\text{def}}{=} \{i\}$ (nullary)
$\mathcal{P} \stackrel{\text{def}}{=} \{R, F\}$ (binary, unary)

Model $\mathcal{M}$:
$A \stackrel{\text{def}}{=} \{a, b, c\}$ (states in computer program)
$i^{\mathcal{M}} \stackrel{\text{def}}{=} a$, $R^{\mathcal{M}} \stackrel{\text{def}}{=} \{(a,a),(a,b),(a,c),(b,c),(c,c)\}$ $F^{\mathcal{M}} \stackrel{\text{def}}{=} \{b, c\}$

2. Informal model check of formula

$$\neg F(i)$$

Example 2.15.
$\mathcal{F} \stackrel{\text{def}}{=} \{i\}$ (nullary)
$\mathcal{P} \stackrel{\text{def}}{=} \{R, F\}$ (binary, unary)

Model $\mathcal{M}$:
$A \stackrel{\text{def}}{=} \{a, b, c\}$ (states in computer program)
$i^{\mathcal{M}} \stackrel{\text{def}}{=} a$, $R^{\mathcal{M}} \stackrel{\text{def}}{=} \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$ $F^{\mathcal{M}} \stackrel{\text{def}}{=} \{b, c\}$

3. Informal model check of formula

$$\forall x \forall y \forall z (R(x, y) \land R(x, z) \to y = z)$$

Example 2.15.
$\mathcal{F} \stackrel{\text{def}}{=} \{i\}$ (nullary)
$\mathcal{P} \stackrel{\text{def}}{=} \{R, F\}$ (binary, unary)

Model $\mathcal{M}$:
$A \stackrel{\text{def}}{=} \{a, b, c\}$ (states in computer program)
$i^{\mathcal{M}} \stackrel{\text{def}}{=} a$, $R^{\mathcal{M}} \stackrel{\text{def}}{=} \{(a,a),(a,b),(a,c),(b,c),(c,c)\}$ $F^{\mathcal{M}} \stackrel{\text{def}}{=} \{b, c\}$

4. Informal model check of formula

$$\forall x \exists y R(x, y)$$

Example 2.16.

$\mathcal{F} \stackrel{\text{def}}{=} \{e, \cdot\}$ (nullary, binary)

$\mathcal{P} \stackrel{\text{def}}{=} \{\leq\}$ (binary)

Infix: $t_1 \cdot t_2 \leq (t \cdot t)$

Model $\mathcal{M}$:

$A \stackrel{\text{def}}{=} \{$(finite) binary strings (including empty string ϵ)$\}$

$e^{\mathcal{M}} \stackrel{\text{def}}{=} \epsilon$

$\cdot^{\mathcal{M}} \stackrel{\text{def}}{=}$ 'concatenation'

$\leq \stackrel{\text{def}}{=}$ 'is prefix'

1. Informal model check of formula

$$\forall x((x \leq x \cdot e) \land (x \cdot e \leq x))$$

Example 2.16.

$\mathcal{F} \stackrel{\text{def}}{=} \{e, \cdot\}$ (nullary, binary)

$\mathcal{P} \stackrel{\text{def}}{=} \{\leq\}$ (binary)

Infix: $t_1 \cdot t_2 \leq (t \cdot t)$

Model $\mathcal{M}$:

$A \stackrel{\text{def}}{=} \{$(finite) binary strings (including empty string ϵ)$\}$

$e^{\mathcal{M}} \stackrel{\text{def}}{=} \epsilon$

$.^{\mathcal{M}} \stackrel{\text{def}}{=}$ ‘concatenation’

$\leq \stackrel{\text{def}}{=}$ ‘is prefix’

2. Informal model check of formula

$$\exists y \forall x (y \leq x)$$

Example 2.16.
$\mathcal{F} \stackrel{\text{def}}{=} \{e, \cdot\}$ (nullary, binary)
$\mathcal{P} \stackrel{\text{def}}{=} \{\leq\}$ (binary)
Infix: $t_1 \cdot t_2 \leq (t \cdot t)$

Model $\mathcal{M}$:
$A \stackrel{\text{def}}{=} \{$(finite) binary strings (including empty string ϵ)$\}$
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Infix: $t_1 \cdot t_2 \leq (t \cdot t)$

Model $\mathcal{M}$:

$A \stackrel{\text{def}}{=} \{$(finite) binary strings (including empty string ϵ)$\}$

$e^{\mathcal{M}} \stackrel{\text{def}}{=} \epsilon$

$.^{\mathcal{M}} \stackrel{\text{def}}{=}$ ‘concatenation’

$\leq \stackrel{\text{def}}{=}$ ‘is prefix’

4. Informal model check of formula

$$\forall x \forall y \forall z ((x \leq y) \rightarrow (x \cdot z \leq y \cdot z))$$

Example 2.16.
$\mathcal{F} \stackrel{\text{def}}{=} \{e, \cdot\}$ (nullary, binary)
$\mathcal{P} \stackrel{\text{def}}{=} \{\leq\}$ (binary)
Infix: $t_1 \cdot t_2 \leq (t \cdot t)$

Model $\mathcal{M}$:
$A \stackrel{\text{def}}{=}$ {(finite) binary strings (including empty string ϵ)}
$e^{\mathcal{M}} \stackrel{\text{def}}{=} \epsilon$
$.^{\mathcal{M}} \stackrel{\text{def}}{=}$ ‘concatenation’
$\leq \stackrel{\text{def}}{=}$ ‘is prefix’

5. Informal model check of formula

$$\neg\exists x \forall y((x \leq y) \rightarrow (y \leq x))$$

Example.

$\mathcal{F} \stackrel{\text{def}}{=} \emptyset$

$\mathcal{P} \stackrel{\text{def}}{=} \{P, Q, R\}$ (unary, unary, binary)

Model $\mathcal{M}$:

$A \stackrel{\text{def}}{=} \{a, b\}$

$P^{\mathcal{M}} \stackrel{\text{def}}{=} \{a, b\}$ $\quad Q^{\mathcal{M}} \stackrel{\text{def}}{=} \{a\}$ $\quad R^{\mathcal{M}} \stackrel{\text{def}}{=} \{(a, a), (a, b)\}$

Informal check of formula

$$\forall x \forall y (P(x) \land \exists x (Q(x) \land R(x, y)))$$

Mild requirements on model...

Choice of model...

$\phi[t/x]$ vs. $\phi[a/x]$

Definition 2.17.

A look-up table or environment for a universe A of concrete values is a function l : **var** $\to A$ from the set of variables **var** to A.

For such an l, we denote by $l[x \mapsto a]$ the look-up table which maps x to a and any other variable y to $l(y)$.

Example.

look-up table l	
x	b
y	b
z	a

updated look-up table $l[x \mapsto a]$	
x	a
y	b
z	a

updated look-up table $l[x \mapsto b]$	
x	b
y	b
z	a

updated look-up table $l[x \mapsto b][x \mapsto a][z \mapsto b]$	
x	a
y	b
z	b

Example.
$\mathcal{F} \stackrel{\text{def}}{=} \emptyset$
$\mathcal{P} \stackrel{\text{def}}{=} \{P, Q, R\}$ (unary, unary, binary)

Model $\mathcal{M}$:
$A \stackrel{\text{def}}{=} \{a, b\}$
$P^{\mathcal{M}} \stackrel{\text{def}}{=} \{a, b\}$ $\quad Q^{\mathcal{M}} \stackrel{\text{def}}{=} \{a\}$ $\quad R^{\mathcal{M}} \stackrel{\text{def}}{=} \{(a, a), (a, b)\}$

What happens to formula

$$\forall x \forall y (P(x) \wedge \exists x (Q(x) \wedge R(x, y)))$$

with

look-up table l	
x	b
y	b

That is: $l(x) = b,\ l(y) = b$

Definition 2.18.
Given a model $\mathcal{M}$ for a pair $(\mathcal{F}, \mathcal{P})$ and given a look-up table l, we define the satisfaction relation $\mathcal{M} \vDash_l \phi$ for each logical formula ϕ over the pair $(\mathcal{F}, \mathcal{P})$ and look-up table l by structural induction on ϕ.

If $\mathcal{M} \vDash_l \phi$ holds, we say that ϕ computes to T in the model $\mathcal{M}$ with respect to the look-up table l.

Definition 2.18. (continued)
P: If ϕ is of the form $P(t_1, t_2, \ldots, t_n)$, then we interpret the terms $t_1, t_2, \ldots, t_n$ in our set A by replacing all variables with their values according to l. In this way we compute concrete values $a_1, a_2, \ldots, a_n$ from A for each of these terms, where we interpret any function symbol $f \in \mathcal{F}$ by $f^{\mathcal{M}}$.
Now $\mathcal{M} \vDash_l P(t_1, t_2, \ldots, t_n)$ holds, iff $(a_1, a_2, \ldots, a_n)$ is in the set $P^{\mathcal{M}}$.

Exercise. Let
$A \stackrel{\text{def}}{=} \{a, b, c\}$
$R^{\mathcal{M}} = \{(b, a), (b, b), (b, c)\}$
$l(x) = b,\ l(y) = c$

(a) Is $\mathcal{M} \vDash_l R(x, y)$?
(b) Is $\mathcal{M} \vDash_l R(y, x)$?

Definition 2.18. (continued)
P: If ϕ is of the form $P(t_1, t_2, \ldots, t_n)$, then we interpret the terms $t_1, t_2, \ldots, t_n$ in our set A by replacing all variables with their values according to l. In this way we compute concrete values $a_1, a_2, \ldots, a_n$ from A for each of these terms, where we interpret any function symbol $f \in \mathcal{F}$ by $f^{\mathcal{M}}$.
Now $\mathcal{M} \models_l P(t_1, t_2, \ldots, t_n)$ holds, iff $(a_1, a_2, \ldots, a_n)$ is in the set $P^{\mathcal{M}}$.

Exercise. Let
$A \stackrel{\text{def}}{=} \{a, b, c\}$
$f^{\mathcal{M}}(a) = f^{\mathcal{M}}(b) = c, \; f^{\mathcal{M}}(c) = b$
$R^{\mathcal{M}} = \{(b, a), (b, b), (b, c)\}$
$l(x) = a, \; l(y) = c$

(a) Is $\mathcal{M} \models_l R(f(x), y)$?
(b) Is $\mathcal{M} \models_l R(f(y), x)$?

Definition 2.18. (continued)

$\forall x$: The relation $\mathcal{M} \vDash_l \forall x \psi$ holds, iff $\mathcal{M} \vDash_{l[x \mapsto a]} \psi$ holds for all $a \in A$.

Exercise. Let

$A \stackrel{\text{def}}{=} \{a, b, c\}$

$R^{\mathcal{M}} = \{(b, a), (b, b), (b, c)\}$

$l(x) = b,\ l(y) = c$

(a) Is $\mathcal{M} \vDash_l \forall x R(x, y)$?

(b) Is $\mathcal{M} \vDash_l \forall y R(x, y)$?

Definition 2.18. (continued)
$\exists x$: The relation $\mathcal{M} \vDash_l \exists x \psi$ holds, iff $\mathcal{M} \vDash_{l[x \mapsto a]} \psi$ holds for some $a \in A$.

Exercise. Let
$A \stackrel{\text{def}}{=} \{a, b, c\}$
$R^{\mathcal{M}} = \{(b, a), (b, b), (b, c)\}$
$l(x) = a,\ l(y) = c$

(a) Is $\mathcal{M} \vDash_l \exists x R(x, y)$?
(b) Is $\mathcal{M} \vDash_l \exists x R(y, x)$?

Definition 2.18. (continued)

$\neg$: The relation $\mathcal{M} \vDash_l \neg\psi$ holds, iff $\mathcal{M} \vDash_l \psi$ does not hold.

$\vee$: The relation $\mathcal{M} \vDash_l \psi_1 \vee \psi_2$ holds, iff $\mathcal{M} \vDash_l \psi_1$ or $\mathcal{M} \vDash_l \psi_2$ holds.

$\wedge$: The relation $\mathcal{M} \vDash_l \psi_1 \wedge \psi_2$ holds, iff $\mathcal{M} \vDash_l \psi_1$ and $\mathcal{M} \vDash_l \psi_2$ holds.

$\rightarrow$: The relation $\mathcal{M} \vDash_l \psi_1 \rightarrow \psi_2$ holds, iff $\mathcal{M} \vDash_l \psi_2$ holds whenever $\mathcal{M} \vDash_l \psi_1$ holds.

Example.

$\mathcal{F} \stackrel{\text{def}}{=} \emptyset$
$\mathcal{P} \stackrel{\text{def}}{=} \{P, Q, R\}$ (unary, unary, binary)

Model $\mathcal{M}$:
$A \stackrel{\text{def}}{=} \{a, b\}$
$P^{\mathcal{M}} \stackrel{\text{def}}{=} \{a, b\}$ $\quad Q^{\mathcal{M}} \stackrel{\text{def}}{=} \{a\}$ $\quad R^{\mathcal{M}} \stackrel{\text{def}}{=} \{(a, a), (a, b)\}$

Is

$$M \vDash_l \forall x \forall y (P(x) \land \exists x (Q(x) \land R(x, y)))$$

with

look-up table l	
x	b
y	b

That is: $l(x) = b,\ l(y) = b$

If l and l' are identical on all free variables in ϕ, then . . .

If ϕ has *no* free variables, then . . .
Notation $\mathcal{M} \vDash \phi$
Sentence ϕ

Example 2.19.
$\mathcal{F} \stackrel{\text{def}}{=} \{\textbf{alma}\}$ (constant)
$\mathcal{P} \stackrel{\text{def}}{=} \{\textbf{loves}\}$ (binary)

Model $\mathcal{M}$:
$A \stackrel{\text{def}}{=} \{a, b, c\}$
$\textbf{alma}^{\mathcal{M}} \stackrel{\text{def}}{=} a$
$\textbf{loves}^{\mathcal{M}} \stackrel{\text{def}}{=} \{(a, a), (b, a), (c, a)\}$

None of Alma's lovers' lovers love her.

In predicate logic: $\phi = \ldots$

Is $M \vDash \phi$?

Example 2.19.
$\mathcal{F} \stackrel{\text{def}}{=} \{\textbf{alma}\}$ (constant)
$\mathcal{P} \stackrel{\text{def}}{=} \{\textbf{loves}\}$ (binary)

Model $\mathcal{M}'$:
$A \stackrel{\text{def}}{=} \{a, b, c\}$
$\textbf{alma}^{\mathcal{M}'} \stackrel{\text{def}}{=} a$
$\textbf{loves}^{\mathcal{M}'} \stackrel{\text{def}}{=} \{(b, a), (c, b)\}$

None of Alma's lovers' lovers love her.

In predicate logic: $\phi = \ldots$

Is $M' \vDash \phi$?