Fundamentele Informatica 3

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Rudy van Vliet
kamer 124 Snellius, tel. 071-527 5777
rvvliet(at)liacs(dot)nl

college 14, 9 mei 2016

10. Computable Functions
10.3. Gödel Numbering
10.4. All Computable Functions are μ-Recursive

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inleverdatum 10 mei 2016, 13:45 uur

A slide from lecture 12:

Theorem 10.4.

Every primitive recursive function is total and computable.

PR:	Turing-computable functions:
total and computable	not necessarily total

A slide from lecture 13:

Definition 10.9. Bounded Quantifications

Let P be an $(n+1)$-place predicate. The *bounded existential quantification* of P is the $(n+1)$-place predicate E_P defined by

$E_P(X,k) =$ (there exists y with $0 \leq y \leq k$ such that $P(X,y)$ is true)

The *bounded universal quantification* of P is the $(n+1)$-place predicate A_P defined by

$$A_P(X,k) = (\text{for every } y \text{ satifying } 0 \leq y \leq k,\ P(X,y) \text{ is true})$$

A slide from lecture 13:

Theorem 10.10.

If P is a primitive recursive $(n+1)$-place predicate,
both the predicates E_P and A_P are also primitive recursive.

Proof. . .

Definition 10.11. Bounded Minimalization

For an $(n+1)$-place predicate P, the *bounded minimalization* of P is the function $m_p : \mathbb{N}^{n+1} \to \mathbb{N}$ defined by

$$m_p(X,k) = \begin{cases} \text{min}\{y \mid \ 0 \le y \le k \text{ and } P(X,y)\} & \text{if this set is not empty} \\ k+1 & \text{otherwise} \end{cases}$$

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The symbol μ is often used for the minimalization operator, and we sometimes write

$$m_P(X,k) = \overset{k}{\mu}\, y[P(X,y)]$$

An important special case is that in which $P(X,y)$ is $(f(X,y) = 0)$, for some $f : \mathbb{N}^{n+1} \to \mathbb{N}$. In this case m_P is written m_f and referred to as the bounded minimalization of f.

A slide from lecture 13:

Exercise.

Let $f : \mathbb{N}^{n+1} \to \mathbb{N}$ be a primitive recursive function.

Show that the predicate $P : \mathbb{N}^{n+1} \to \{\text{true}, \text{false}\}$ defined by

$$P(X, y) = \ (f(X, y) = 0)$$

is primitive recursive.

Theorem 10.12.

If P is a primitive recursive $(n+1)$-place predicate,
its bounded minimalization m_P is a primitive recursive function.

Proof. . .

$$h(X,y,z) = \begin{cases} z & \text{if } z \leq y \\ y+1 & \text{if } z \geq y+1 \ \wedge \ P(X,y+1) \text{ is true} \\ y+2 & \text{if } z \geq y+1 \ \wedge \ \neg P(X,y+1) \text{ is true} \end{cases}$$

$$h(X,y,z) = \begin{cases} z & \text{if } E_P(X,y) \text{ is true} \\ y+1 & \text{if } \neg E_P(X,y) \ \wedge \ P(X,y+1) \text{ is true} \\ y+2 & \text{if } \neg E_P(X,y) \ \wedge \ \neg P(X,y+1) \text{ is true} \end{cases}$$

Example 10.13. The nth Prime Number

$$\mathit{PrNo}(0) = 2$$
$$\mathit{PrNo}(1) = 3$$
$$\mathit{PrNo}(2) = 5$$

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$PrNo(2) = 5$

$$\begin{aligned} Prime(n) &= (n \geq 2) \wedge \neg(\text{there exists } y \text{ such that} \\ &\qquad y \geq 2 \wedge y \leq n - 1 \wedge Mod(n, y) = 0) \end{aligned}$$

Example 10.13. The nth Prime Number

Let

$$P(x,y) = \ (y > x \wedge \textit{Prime}(y))$$

Then $m_P(x,k)$ ...
and

$$\begin{aligned} \textit{PrNo}(0) &= 2 \\ \textit{PrNo}(k+1) &= m_P(\textit{PrNo}(k), (\textit{PrNo}(k))! + 1) \end{aligned}$$

is primitive recursive, with $h(x_1, x_2) = \ldots$

A slide from lecture 12:

Theorem 10.4.

Every primitive recursive function is total and computable.

PR:
total and computable

Turing-computable functions:
not necessarily total

Unbounded minimalization

Total?

Unbounded minimalization

Total?

A possible definition:

$$M(X) = \begin{cases} (\min\{y \mid \ P(X,y) \text{ is true}\}) + 1 & \text{if this set is not empty} \\ 0 & \text{otherwise} \end{cases}$$

Computable?

A slide from lecture 13:

(Un)bounded quantification

$H(x, y) =$ T_u halts after exactly y moves on input s_x

Halts$(x) =$ there exists y such that
T_u halts after exactly y moves on input s_x

Definition 10.14. Unbounded Minimalization

If P is an $(n+1)$-place predicate, the *unbounded minimalization* of P is the partial function $M_P : \mathbb{N}^n \to \mathbb{N}$ defined by

$$M_P(X) = \text{min}\{y \mid \ P(X, y) \text{ is true}\}$$

$M_P(X)$ is undefined at any $X \in \mathbb{N}^n$ for which there is no y satisfying $P(X, y)$.

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$M_P(X)$ is undefined at any $X \in \mathbb{N}^n$ for which there is no y satisfying $P(X,y)$.

The notation $\mu\, y[P(X,y)]$ is also used for $M_P(X)$.
In the special case in which $P(X,y) = (f(X,y) = 0)$, we write $M_P = M_f$ and refer to this function as the unbounded minimalization of f.

Definition 10.15. μ-Recursive Functions

The set $\mathcal{M}$ of μ-recursive, or simply *recursive*, partial functions is defined as follows.

1. Every initial function is an element of $\mathcal{M}$.

2. Every function obtained from elements of $\mathcal{M}$ by composition or primitive recursion is an element of $\mathcal{M}$.

3. For every $n \geq 0$ and every total function $f : \mathbb{N}^{n+1} \to \mathbb{N}$ in $\mathcal{M}$, the function $M_f : \mathbb{N}^n \to \mathbb{N}$ defined by
$$M_f(X) = \mu\, y[f(X, y) = 0]$$
is an element of $\mathcal{M}$.

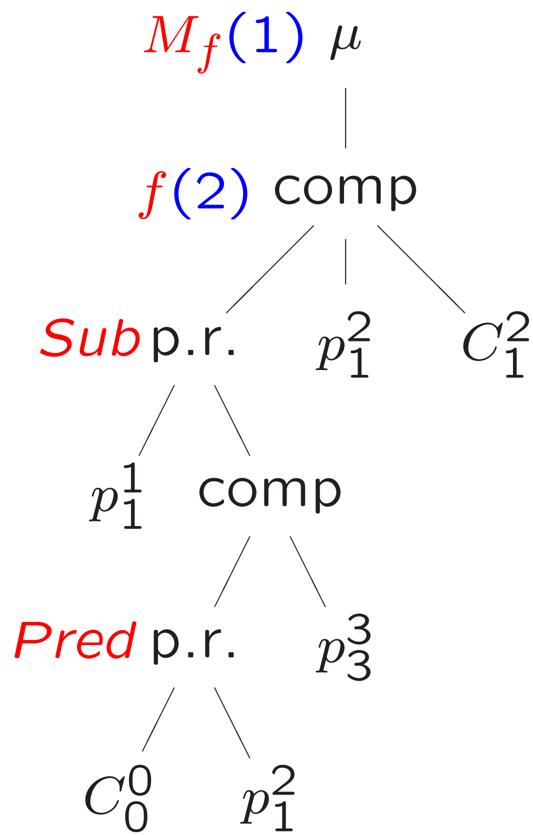
Example.

Let

$$f(x,k) = p_1^2(x,k) \dot{-} C_1^2(x,k)$$

$M_f(x)$ . . .

Structure tree M_f:



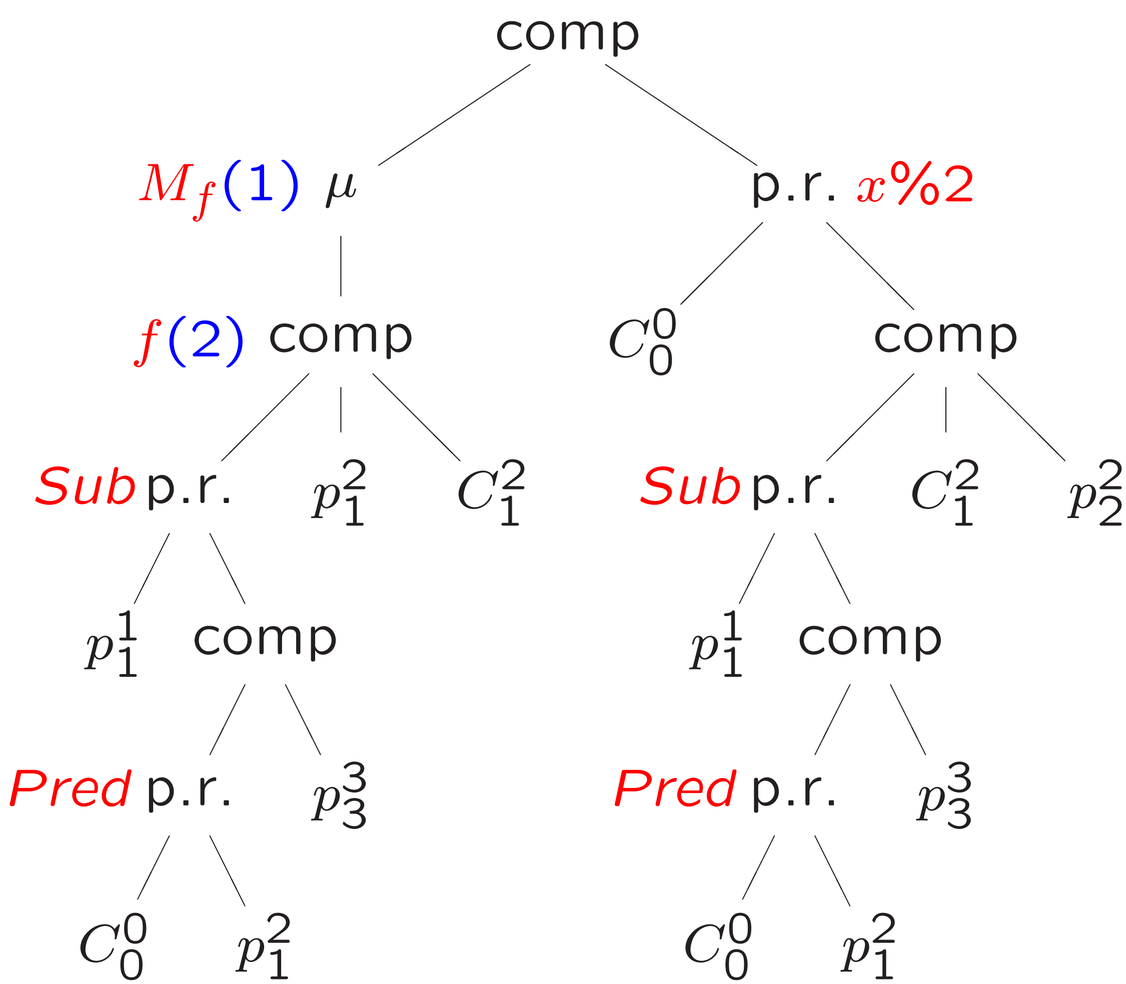
Not total

Exercise.

a. Give an example of a non-total function f and another function g, such that the composition of f and g is total.

b. Can you also find an example of a non-total function f and another function g, such that the composition of g and f is total?

Structure tree $M_f(x\%2)$:



Total

Theorem 10.16.

All μ-recursive partial functions are computable.

Proof. . .

10.3. Gödel Numbering

Definition 10.17.
The Gödel Number of a Sequence of Natural Numbers

For every $n \geq 1$ and every finite sequence $x_0, x_1, \ldots, x_{n-1}$ of n natural numbers, the *Gödel number* of the sequence is the number

$$\mathit{gn}(x_0, x_1, \ldots, x_{n-1}) = 2^{x_0} 3^{x_1} 5^{x_2} \ldots (\mathit{PrNo}(n-1))^{x_{n-1}}$$

where *PrNo*(i) is the ith prime (Example 10.13).

Exercise 10.16.

Show that for any $n \geq 1$, the functions Add_n and Mult_n from $\mathbb{N}^n$ to $\mathbb{N}$, defined by

$$\begin{aligned} \mathit{Add}_n(x_1, \ldots, x_n) &= x_1 + x_2 + \cdots + x_n \\ \mathit{Mult}_n(x_1, \ldots, x_n) &= x_1 * x_2 * \cdots * x_n \end{aligned}$$

respectively, are both primitive recursive.

Example 10.18.

The Power to Which a Prime is Raised in the Factorization of x

Function *Exponent* : $\mathbb{N}^2 \to \mathbb{N}$ defined as follows:

$$\textit{Exponent}(i, x) = \begin{cases} \text{the exp. of } \textit{PrNo}(i) \text{ in } x\text{'s prime fact.} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

A slide from lecture 12

Definition 10.2. The Operations of Composition and Primitive Recursion (continued)

2. Suppose $n \geq 0$ and g and h are functions of n and $n+2$ variables, respectively. (By "a function of 0 variables," we mean simply a constant.)
   The function obtained from g and h by the operation of *primitive recursion* is the function $f : \mathbb{N}^{n+1} \to \mathbb{N}$ defined by the formulas

$$\begin{aligned} f(X, 0) &= g(X) \\ f(X, k+1) &= h(X, k, f(X, k)) \end{aligned}$$

   for every $X \in \mathbb{N}^n$ and every $k \geq 0$.

$$f(X, k+1) = h(X, k, f(X, 0), \ldots, f(X, k))$$

$$(f(X,0),\ldots,f(X,k),f(X,k+1)) = h(X,k,(f(X,0),\ldots,f(X,k)))$$

Theorem 10.19.

Suppose that $g : \mathbb{N}^n \to \mathbb{N}$ and $h : \mathbb{N}^{n+2} \to \mathbb{N}$ are primitive recursive functions, and $f : \mathbb{N}^{n+1} \to \mathbb{N}$ is obtained from g and h by course-of-values recursion; that is

$$\begin{aligned} f(X,0) &= g(X) \\ f(X,k+1) &= h(X,k,\mathit{gn}(f(X,0),\ldots,f(X,k))) \end{aligned}$$

Then f is primitive recursive.

Proof. . .

Example.

Fibonacci

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f(n-1) + f(n-2) & \text{if } n \geq 2 \end{cases}$$