Fundamentele Informatica 3

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10. Computable Functions
10.2. Quantification, Minimalization, and μ-Recursive Functions
10.3. Gödel Numbering

A slide from lecture 13

Definition 10.9. Bounded Quantifications

Let P be an $(n+1)$-place predicate. The *bounded existential quantification* of P is the $(n+1)$-place predicate E_P defined by

$E_P(X,k) =$ (there exists y with $0 \leq y \leq k$ such that $P(X,y)$ is true)

The *bounded universal quantification* of P is the $(n+1)$-place predicate A_P defined by

$$A_P(X,k) = \text{(for every } y \text{ satifying } 0 \leq y \leq k,\ P(X,y) \text{ is true)}$$

A slide from lecture 13

Theorem 10.10.

If P is a primitive recursive $(n+1)$-place predicate,
both the predicates E_P and A_P are also primitive recursive.

Proof. . .

A slide from lecture 13

Definition 10.11. Bounded Minimalization

For an $(n+1)$-place predicate P, the *bounded minimalization* of P is the function $m_P : \mathbb{N}^{n+1} \to \mathbb{N}$ defined by

$$m_P(X,k) = \begin{cases} \text{min}\{y \mid \ 0 \le y \le k \text{ and } P(X,y)\} & \text{if this set is not empty} \\ k+1 & \text{otherwise} \end{cases}$$

The symbol μ is often used for the minimalization operator, and we sometimes write

$$m_P(X,k) = \overset{k}{\mu}\, y[P(X,y)]$$

An important special case is that in which $P(X,y)$ is $(f(X,y) = 0)$, for some $f : \mathbb{N}^{n+1} \to \mathbb{N}$. In this case m_P is written m_f and referred to as the bounded minimalization of f.

A slide from lecture 13

Theorem 10.12.

If P is a primitive recursive $(n+1)$-place predicate,
its bounded minimalization m_P is a primitive recursive function.

Proof. . .

Example 10.13. The nth Prime Number

$$PrNo(0) = 2$$
$$PrNo(1) = 3$$
$$PrNo(2) = 5$$

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$$\begin{aligned} Prime(n) &= (n \geq 2) \land \neg(\text{there exists } y \text{ such that} \\ &\qquad y \geq 2 \land y \leq n-1 \land Mod(n,y) = 0) \end{aligned}$$

Example 10.13. The nth Prime Number

Let

$$P(x, y) = (y > x \wedge \mathit{Prime}(y))$$

Then $m_P(x, k)$ ...
and

$$\begin{aligned} \mathit{PrNo}(0) &= 2 \\ \mathit{PrNo}(k+1) &= \ldots \end{aligned}$$

Example 10.13. The nth Prime Number

Let

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Then $m_P(x,k)$ ...
and

$$\begin{aligned} \mathit{PrNo}(0) &= 2 \\ \mathit{PrNo}(k+1) &= m_P(\mathit{PrNo}(k), (\mathit{PrNo}(k))! + 1) \end{aligned}$$

is primitive recursive, with $h(x_1, x_2) = \ldots$

A slide from lecture 9

Application:

```
n = 4;
while (n is the sum of two primes)
  n = n+2;
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This program loops forever, if and only if Goldbach's conjecture is true.

Exercise 10.19.

Show that each of the following functions is primitive recursive.

b. $f : \mathbb{N}^2 \to \mathbb{N}$ defined by $f(x, y) = \min\{x, y\}$

c. $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = \lfloor \sqrt{x} \rfloor$
(the largest natural number less than or equal to $\sqrt{x}$)

d. $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = \lfloor \log_2(x + 1) \rfloor$

Exercise 10.23.

In addition to the bounded minimalization of a predicate, we might define the bounded maximalization of a predicate P to be the function m^P defined by

$$m^P(X,k) = \begin{cases} \text{max}\{y \leq k \mid \ P(x,y) \text{ is true}\} & \text{if this set is not empty} \\ 0 & \text{otherwise} \end{cases}$$

a. Show m^P is primitive recursive by finding two primitive recursive functions from which it can be obtained by primitive recursion.

b. Show m^P is primitive recursive by using bounded minimalization.

A slide from lecture 12

Theorem 10.4.

Every primitive recursive function is total and computable.

PR:
total and computable

Turing-computable functions:
not necessarily total

Unbounded minimalization

Total?

Unbounded minimalization

Total?

A possible definition:

$$M(X) = \begin{cases} (\min\{y \mid \ P(X, y) \text{ is true}\}) + 1 & \text{if this set is not empty} \\ 0 & \text{otherwise} \end{cases}$$

Computable?

A slide from lecture 13

(Un)bounded quantification

$H(x, y) =$ T_u halts after exactly y moves on input s_x

Halts$(x) =$ there exists y such that
T_u halts after exactly y moves on input s_x

Definition 10.14. Unbounded Minimalization

If P is an $(n+1)$-place predicate, the *unbounded minimalization* of P is the partial function $M_P : \mathbb{N}^n \to \mathbb{N}$ defined by

$$M_P(X) = \min\{y \mid \ P(X, y) \text{ is true}\}$$

$M_P(X)$ is undefined at any $X \in \mathbb{N}^n$ for which there is no y satisfying $P(X, y)$.

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The notation $\mu\, y[P(X,y)]$ is also used for $M_P(X)$.
In the special case in which $P(X,y) = (f(X,y) = 0)$, we write $M_P = M_f$ and refer to this function as the unbounded minimalization of f.

Exercise 10.30.

Show that the unbounded minimalization of any predicate can be written in the form $\mu\, y[f(X, y) = 0]$, for some function f.

Definition 10.15. μ-Recursive Functions

The set $\mathcal{M}$ of μ-recursive, or simply *recursive*, partial functions is defined as follows.

1. Every initial function is an element of $\mathcal{M}$.

2. Every function obtained from elements of $\mathcal{M}$ by composition or primitive recursion is an element of $\mathcal{M}$.

3. For every $n \geq 0$ and every total function $f : \mathbb{N}^{n+1} \to \mathbb{N}$ in $\mathcal{M}$, the function $M_f : \mathbb{N}^n \to \mathbb{N}$ defined by
$$M_f(X) = \mu\, y[f(X, y) = 0]$$
is an element of $\mathcal{M}$.

In particular, f may be any primitive recursive function.

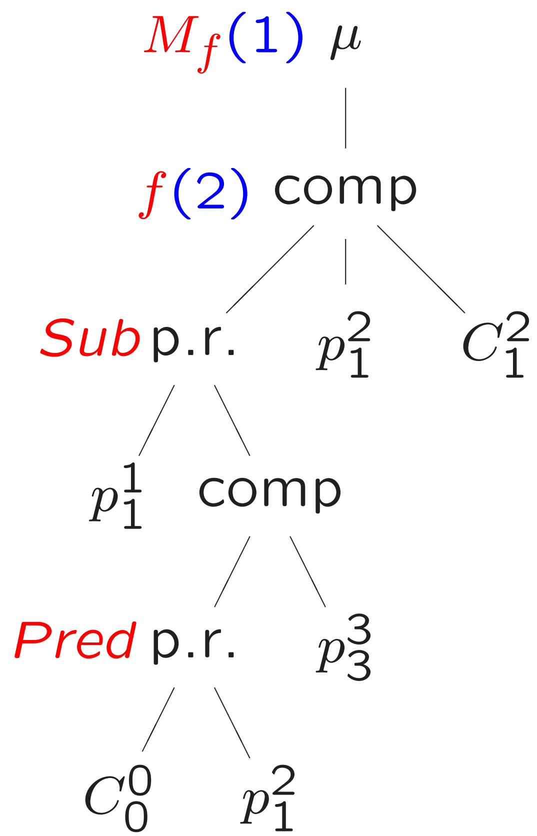
Example.

Let

$$f(x, y) = p_1^2(x, y) \dot{-} C_1^2(x, y)$$

$M_f(x)$ . . .

Structure tree M_f:



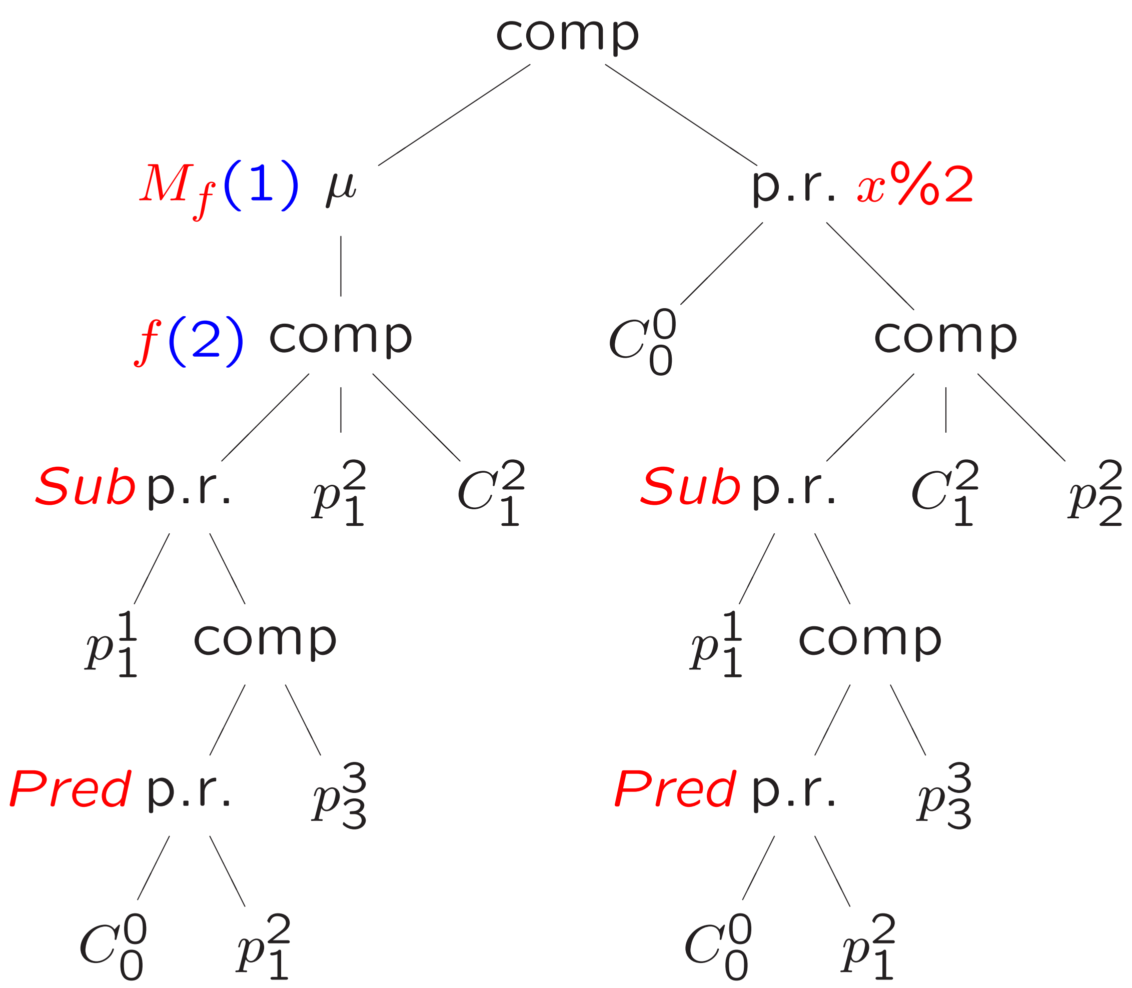
Not total

Exercise.

a. Give an example of a non-total function f and another function g, such that the composition of f and g is total.

b. Can you also find an example of a non-total function f and another function g, such that the composition of g and f is total?

Structure tree $M_f(x\%2)$:



Total

Theorem 10.16.

All μ-recursive partial functions are computable.

Proof. . .

10.3. Gödel Numbering

Definition 10.17.
The Gödel Number of a Sequence of Natural Numbers

For every $n \geq 1$ and every finite sequence $x_0, x_1, \ldots, x_{n-1}$ of n natural numbers, the *Gödel number* of the sequence is the number

$$\mathit{gn}(x_0, x_1, \ldots, x_{n-1}) = 2^{x_0} 3^{x_1} 5^{x_2} \ldots (\mathit{PrNo}(n-1))^{x_{n-1}}$$

where $\mathit{PrNo}(i)$ is the ith prime (Example 10.13).

Exercise 10.16.

Show that for any $n \geq 1$, the functions Add_n and Mult_n from $\mathbb{N}^n$ to $\mathbb{N}$, defined by

$$\begin{aligned} \mathit{Add}_n(x_1, \ldots, x_n) &= x_1 + x_2 + \cdots + x_n \\ \mathit{Mult}_n(x_1, \ldots, x_n) &= x_1 * x_2 * \cdots * x_n \end{aligned}$$

respectively, are both primitive recursive.

Example 10.18.

The Power to Which a Prime is Raised in the Factorization of x

Function *Exponent* : $\mathbb{N}^2 \to \mathbb{N}$ defined as follows:

$$\textit{Exponent}(i, x) = \begin{cases} \text{the exp. of } \textit{PrNo}(i) \text{ in } x\text{'s prime fact.} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$