Fundamentele Informatica 3

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9. Undecidable Problems
9.5. Undecidable Problems
Involving Context-Free Languages

Huiswerkopgave 3

Reducties en (on-)beslisbaarheid

A slide from lecture 9

Definition 9.6. Reducing One Decision Problem to Another, and Reducing One Language to Another

Suppose P_1 and P_2 are decision problems. We say P_1 is reducible to P_2 ($P_1 \leq P_2$)
- if there is an algorithm
- that finds, for an arbitrary instance I of P_1, an instance $F(I)$ of P_2,
- such that

  for every I the answers for the two instances are the same,
  or I is a yes-instance of P_1

  if and only if $F(I)$ is a yes-instance of P_2.

A slide from lecture 9

Theorem 9.7.

...

Suppose P_1 and P_2 are decision problems, and $P_1 \leq P_2$. If P_2 is decidable, then P_1 is decidable.

A slide from lecture 10

9.4. Post's Correspondence Problem

Instance:

10	01	0	100	1
101	100	10	0	010

A slide from lecture 10

Instance:

10	01	0	100	1
101	100	10	0	010

Match:

10	1	01	0	100	100	0	100
101	010	100	10	0	0	10	0

A slide from lecture 10

Definition 9.14. Post's Correspondence Problem

An instance of Post's correspondence problem (*PCP*) is a set

$$\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$$

of pairs, where $n \geq 1$ and the α_i's and β_i's are all nonnull strings over an alphabet Σ.

The decision problem is this:

Given an instance of this type, do there exist a positive integer k and a sequence of integers $i_1, i_2, \ldots, i_k$, with each i_j satisfying $1 \leq i_j \leq n$, satisfying

$$\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_k} = \beta_{i_1}\beta_{i_2}\ldots\beta_{i_k} \quad ?$$

$i_1, i_2, \ldots, i_k$ need not all be distinct.

A slide from lecture 10

Theorem 9.17.
Post's correspondence problem is undecidable.

9.5. Undecidable Problems Involving Context-Free Languages

For an instance

$$\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$$

of *PCP*, let...

CFG G_α be defined by productions...

For an instance

$$\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$$

of PCP, let. . .

CFG G_α be defined by productions

$$S_\alpha \to \alpha_i S_\alpha c_i \mid \alpha_i c_i \quad (1 \leq i \leq n)$$

Example derivation:

$S_\alpha \Rightarrow \alpha_2 S_\alpha c_2 \Rightarrow \alpha_2 \alpha_5 S_\alpha c_5 c_2 \Rightarrow \alpha_2 \alpha_5 \alpha_1 S_\alpha c_1 c_5 c_2 \Rightarrow \alpha_2 \alpha_5 \alpha_1 \alpha_3 c_3 c_1 c_5 c_2$

Unambiguous

For an instance

$$\{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$$

of *PCP*, let...

CFG G_α be defined by productions

$$S_\alpha \to \alpha_i S_\alpha c_i \mid \alpha_i c_i \quad (1 \leq i \leq n)$$

CFG G_β be defined by productions

$$S_\beta \to \beta_i S_\beta c_i \mid \beta_i c_i \quad (1 \leq i \leq n)$$

Example.

Let I be the following instance of PCP:

10	01	0	100	1
101	100	10	0	010

G_α and G_β. . .

Theorem 9.20.
These two problems are undecidable:

1. *CFGNonEmptyIntersection*:
   Given two CFGs G_1 and G_2, is $L(G_1) \cap L(G_2)$ nonempty?

2. *IsAmbiguous*:
   Given a CFG G, is G ambiguous?

Proof. . .

Theorem 9.20.
This problem is undecidable:

1. *CFGNonEmptyIntersection*:
   Given two CFGs G_1 and G_2, is $L(G_1) \cap L(G_2)$ nonempty?

Alternative proof. . .

Let CFG G_1 be defined by productions

$$S_1 \to \alpha_i S_1 \beta_i^r \;\mid\; \alpha_i \# \beta_i^r \quad (1 \le i \le n)$$

Let CFG G_2 be defined by productions

$$S_2 \to a S_2 a \;\mid\; b S_2 b \;\mid\; a \# a \;\mid\; b \# b$$

Let T be TM, let x be string accepted by T, and let

$$z_0 \vdash z_1 \vdash z_2 \vdash z_3 \ldots \vdash z_n$$

be 'successful computation' of T for x,
i.e., $z_0 = q_0 \Delta x$
and z_n is accepting configuration.

Let T be TM, let x be string accepted by T, and let

$$z_0 \vdash z_1 \vdash z_2 \vdash z_3 \ldots \vdash z_n$$

be 'successful computation' of T for x,
i.e., $z_0 = q_0 \Delta x$
and z_n is accepting configuration.

Successive configurations z_i and z_{i+1} are almost identical; hence the language

$$\{z\#z'\# \mid \ z \text{ and } z' \text{ are config's of } T \text{ for which } z \vdash z'\}$$

cannot be described by CFG,
cf. $XX = \{xx \mid x \in \{a,b\}^*\}$.

Let T be TM, let x be string accepted by T, and let

$$z_0 \vdash z_1 \vdash z_2 \vdash z_3 \ldots \vdash z_n$$

be 'successful computation' of T for x,
i.e., $z_0 = q_0\Delta x$
and z_n is accepting configuration.

On the other hand, $z_i\#z_{i+1}^r$ is almost a palindrome, and palindromes *can* be described by CFG.

Lemma.

The language

$$L_1 = \{z\#(z')^r\# \mid z \text{ and } z' \text{ are config's of } T \text{ for which } z \vdash z'\}$$

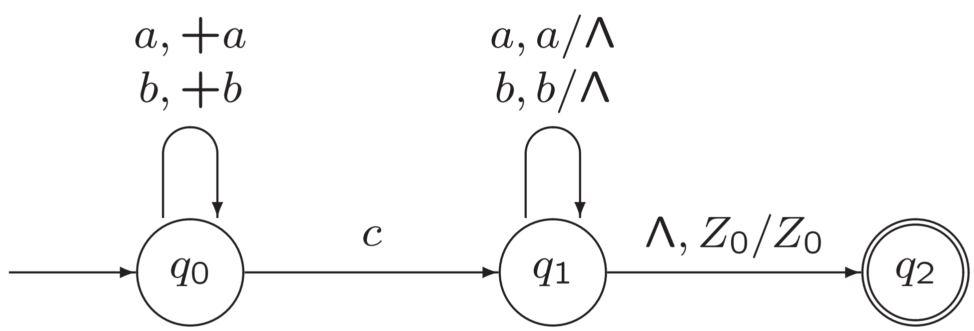
is context-free.

Proof. . .

A slide from lecture 1

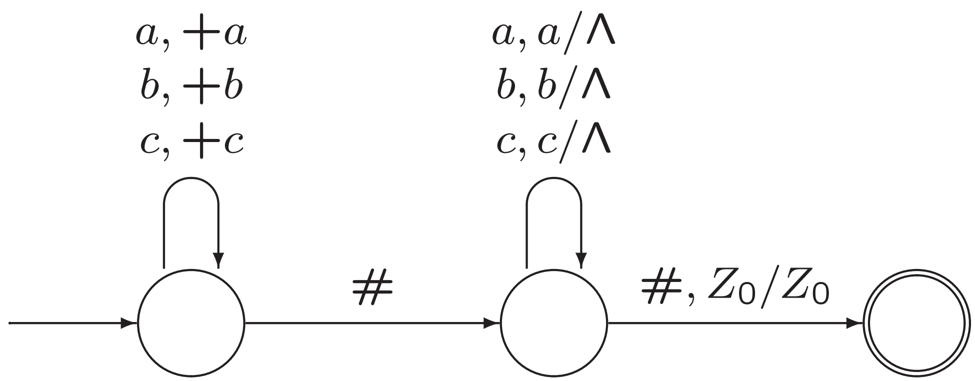
Example 5.3. A Pushdown Automaton Accepting *SimplePal*

SimplePal $= \{xcx^r \mid x \in \{a,b\}^*\}$



A Pushdown Automaton Accepting L_0

$L_0 = \{z\#z^r\# \mid z \in \{a,b,c\}^*\}$



Now adjust for

$L_1 \;=\; \{z\#(z')^r\# \mid$ z and z' are config's of T for which $z \vdash z'\}$

Definition 9.21. Valid Computations of a TM

Let $T = (Q, \Sigma, \Gamma, q_0, \delta)$ be a Turing machine.

A *valid computation* of T is a string of the form

$$z_0 \# z_1^r \# z_2 \# z_3^r \ldots \# z_n \#$$

if n is even, or

$$z_0 \# z_1^r \# z_2 \# z_3^r \ldots \# z_n^r \#$$

if n is odd,
where in either case, $\#$ is a symbol not in Γ,
and the strings z_i represent successive configurations of T on some input string x, starting with the initial configuration z_0 and ending with an accepting configuration.

The set of valid computations of T will be denoted by C_T.

Theorem 9.22.

For a TM $T = (Q, \Sigma, \Gamma, q_0, \delta)$,
• the set C_T of valid computations of T is the intersection of two context-free languages,
• and its complement C'_T is a context-free language.

Proof. . .

Theorem 9.22.

For a TM $T = (Q, \Sigma, \Gamma, q_0, \delta)$,
• the set C_T of valid computations of T is the intersection of two context-free languages,
• and its complement C'_T is a context-free language.

Proof. Let

$$
\begin{aligned}
L_1 &= \{z\#(z')^r\# \mid \ z \text{ and } z' \text{ are config's of } T \text{ for which } z \vdash z'\} \\
L_2 &= \{z^r\#z'\# \mid \ z \text{ and } z' \text{ are config's of } T \text{ for which } z \vdash z'\} \\
I &= \{z\# \mid \ z \text{ is initial configuration of } T\} \\
A &= \{z\# \mid \ z \text{ is accepting configuration of } T\} \\
A_1 &= \{z^r\# \mid \ z \text{ is accepting configuration of } T\}
\end{aligned}
$$

$$C_T = L_3 \cap L_4$$

where

$$\begin{aligned} L_3 &= I L_2^*(A_1 \cup \{\Lambda\}) \\ L_4 &= L_1^*(A \cup \{\Lambda\}) \end{aligned}$$

for each of which we can algorithmically construct a CFG

If $x \in C'_T$ (i.e., $x \notin C_T$), then. . .

Definition 9.21. Valid Computations of a TM

Let $T = (Q, \Sigma, \Gamma, q_0, \delta)$ be a Turing machine.

A *valid computation* of T is a string of the form

$$z_0 \# z_1^r \# z_2 \# z_3^r \ldots \# z_n \#$$

if n is even, or

$$z_0 \# z_1^r \# z_2 \# z_3^r \ldots \# z_n^r \#$$

if n is odd,
where in either case, $\#$ is a symbol not in Γ,
and the strings z_i represent successive configurations of T on some input string x, starting with the initial configuration z_0 and ending with an accepting configuration.

The set of valid computations of T will be denoted by C_T.

If $x \in C'_T$ (i.e., $x \notin C_T$), then

1. Either, x does not end with #

Otherwise, let $x = z_0 \# z_1 \# \ldots \# z_n \#$

(no reversed strings in this partitioning)

2. Or, for some even i, z_i is not configuration of T

3. Or, for some odd i, z_i^r is not configuration of T

4. Or z_0 is not initial configuration of T

5. Or z_n is neither accepting configuration, nor the reverse of one

6. Or, for some even i, $z_i \not\vdash z_{i+1}^r$

7. Or, for some odd i, $z_i^r \not\vdash z_{i+1}$

If $x \in C'_T$ (i.e., $x \notin C_T$), then

1. Either, x does not end with #

Otherwise, let $x = z_0\#z_1\#\ldots\#z_n\#$

2. Or, for some even i, z_i is not configuration of T
3. Or, for some odd i, z_i^r is not configuration of T
4. Or z_0 is not initial configuration of T
5. Or z_n is neither accepting configuration, nor the reverse of one
6. Or, for some even i, $z_i \not\vdash z_{i+1}^r$
7. Or, for some odd i, $z_i^r \not\vdash z_{i+1}$

Hence, C'_T is union of seven context-free languages,
for each of which we can algorithmically construct a CFG

Corollary.

The decision problem

> *CFGNonEmptyIntersection*:
> Given two CFGs G_1 and G_2, is $L(G_1) \cap L(G_2)$ nonempty?

is undecidable (cf. Theorem 9.20(1)).

Proof.

Let
AcceptsSomething: Given a TM T, is $L(T) \neq \emptyset$?

Prove that *AcceptsSomething* $\leq$ *CFGNonEmptyIntersection*

Theorem 9.23. The decision problem

> *CFGGeneratesAll*: Given a CFG G with terminal alphabet Σ, is $L(G) = \Sigma^*$?

is undecidable.

Proof.

Let
AcceptsNothing: Given a TM T, is $L(T) = \emptyset$?

Prove that *AcceptsNothing* $\leq$ *CFGGeneratesAll* . . .

Undecidable Decision Problems (we have discussed)