Corrections of C.F.I. 5th of January 1996

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- We consider only *reversible* 2-structures.
- Please always explain your answers, and in particular quote the results you use to arrive at a given answer.

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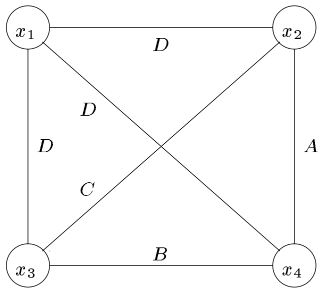
1. Let $g = (D, \mathcal{P}) = (D, R)$ be a 2s and $X, Y \in \mathcal{C}(g)$.

   (i) Prove that if X and Y are disjoint then there is an equivalence class $K \in \mathcal{P}$ such that for all $x \in X$ and $y \in Y$: $(x, y) \in K$.
   Answer: if either of X and Y is empty or a singleton then the statement is trivially true. Assume then that $|X|, |Y| > 1$. Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ be arbitrary nodes and let $(x_1, y_1) \in K$. Because Y is a clan $(x_1, y_2) \in K$. Then, because X is a clan $(x_2, y_2) \in K$. $\square$
   Is the disjointness condition necessary? If not, prove it; otherwise give a counterexample.
   Answer: the disjointness condition is indeed necessary. A counterexample is given below. In this example $X = \{x_2, x_3, x_4\}$ and $Y = \{x_2\}$ are clans. (The labels indicate the equivalence classes of the 2s.) On the one hand there is no edge from every $x \in X$ to every $y \in Y$, but this is rather trivial and the proposition could be easily changed to compensate for this. If we do this we can still note that (x_2, x_3) and (x_2, x_4) are in different classes.



(ii) Prove that if $X \cap Y \neq \emptyset$ then $X \cup Y \in \mathcal{C}(g)$.
Answer: the trival cases occur when $X \subseteq Y$ or $Y \subseteq X$. In the first case $X \cup Y$ equals Y and so is a clan; in the second case $X \cup Y = X$ and so is a clan. The non trivial case is when X and Y overlap. Let $u \in X - Y$, $w \in Y - X$ and $v \in X \cap Y$. Now, let $y \in D - (X \cup Y)$. Because X is a clan $(y, u)R(y, v)$ and because Y is a clan $(y, v)R(y, w)$. Transitivity of R implies that $(y, u)R(y, w)$ and hence $X \cup Y$ is a clan. □
Is the nonemptiness of the intersection necessary? If not, prove it; otherwise give a counterexample.
Answer: the nonemptiness condition is indeed necessary. Every singleton is a clan and if the theorem would hold in general all subsets of the domain would be clans, which is not true in general for a 2s.

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2. (i) Give the definition of a *linear 2-structure.*
Answer: quoted from the reader:
A 2s $g = (D, \mathcal{P})$ is linear iff $|D| = 1$ or g is antisymmetric, $\mathcal{P} = \{P, P'\}$, and there exists a linear order $x_1, \ldots, x_n$ of D such that for all different $i, j \in \{1, \ldots, n\}$: $(x_i, x_j) \in P$ iff $i < j$.

(ii) Give a characterization of the clans of a linear 2-structure in terms of segments of linear orders. Prove it.
Answer: recall that a segment S is a subset of D such that all nodes in S are directly subsequent in the linear order in the definition above. (Note that here a segment can contain any number of elements; the reader implies that a segment has at least two elements and less than $|D|$.) The characterization is simply that the set of segments of g (denoted $\mathrm{SEG}(g)$) equals $\mathcal{C}(g)$. This must of course be proven. We prove this by proving the two inclusions. First of all, a segment is a clan. This can be proven by saying that every element before a segment in the linear order sees every element in the segment the same way (all with an edge in P, from the definition in part (i)) and all elements after the segment see all elements also in the same way (namely with an edge from P'). Hence, the segment is a clan.
The other way around is also not too difficult. Let X be a clan that is not a segment. Still, there is a shortest segment S such that $X \subseteq S$ and this inclusion is proper, because X is not a segment. For some $s \in S - X$ some edges from s going into X are in P and some are in P'. Hence, X is not a clan. Contradiction. The two inclusions imply that the set of segments equals the set of clans. □

In the following exercise we have given the answers for partial 2-structures and not for labeled partial 2-structures. The definitions for the latter are the ones from the reader. Both versions are correct, as the difference is not really material.

3. (i) Give the definitions of a morphism, a uniform morphism and an isomorphism between ℓp2s's.

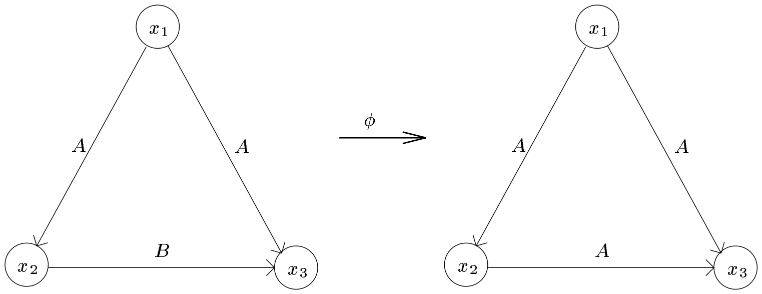
Answer: let $g_i = (D_i, F_i, R_i)$ for $i = 1, 2$. Let ϕ be a mapping from D_1 to D_2. Then ϕ is a *morphism* from g_1 to g_2 if for all g_1-equivalent $(x, y), (u, v)$ in F_1: if $\phi(x) \neq \phi(y)$ and $\phi(u) \neq \phi(v)$ then $(\phi(x), \phi(y)), (\phi(u), \phi(v)) \in F_2$ and $(\phi(x), \phi(y)) R_2 (\phi(u), \phi(v))$.

And ϕ is a uniform morphism if ϕ is a morphism and for all g_1-equivalent $(x, y), (u, v)$ in F_1: $\phi(x) = \phi(y)$ iff $\phi(u) = \phi(v)$.

Lastly, ϕ is a unimorphism if ϕ is a morphism that is bijective and the inverse of ϕ is also a morphism.

(ii) Give an example of two ℓp2s's and a morphism ϕ between them, such that ϕ is *not* an isomorphism.

Answer: an example is given in the figure below.



Let ϕ be the identity function on D_1. It is easy to see that ϕ is a morphism because it does not implode any nodes and all A labeled edges stay in the same class. However they are joined by the B labeled edge and hence the inverse of ϕ is not a morphism (although ϕ is bijective).

(iii) Prove that if ϕ is a uniform morphism from g_1 to g_2 then for all $W \in R_{g_2}$, $\phi^{-1}(W) \in R_{g_1}$.

Answer: let ϕ be a uniform morphism from g_1 to g_2, let $W \in R_{g_2}$ and let $W' = \phi^{-1}(W) = \{x \mid \phi(x) \in W\}$. Also let $D = \underline{\mathrm{dom}}(g_1)$. Assume W' is a not a region of g_1. We will prove that W cannot be a region of g_2.

If W' is not a region of g_1 then we have two possibilities that are complementary (we will prove only one). For two different 2-edges (x, y) and (u, v) of D with $(x, y) R_{g_1} (u, v)$, it holds that either (x, y) is leaving and (u, v) is not or (x, y) is entering and (u, v) is not.

We should state first that $\phi(x) \neq \phi(y)$ and thus by uniformity $\phi(u) \neq \phi(v)$. This holds, because if $z = \phi(x) = \phi(y)$, $z \in W$ and thus $x, y \in W'$ or $z \notin W$ and then $x, y \notin W'$. Then, by definition of morphism, $(\phi(x), \phi(y)) R_{g_2} (\phi(u), \phi(v))$. Hence, $W \notin R_{g_2}$.

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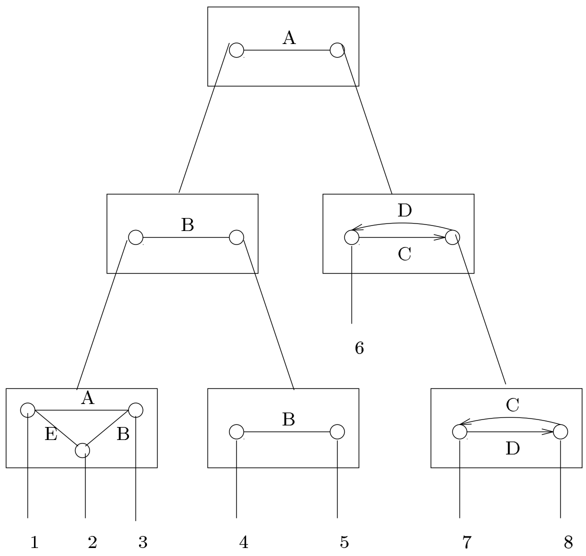
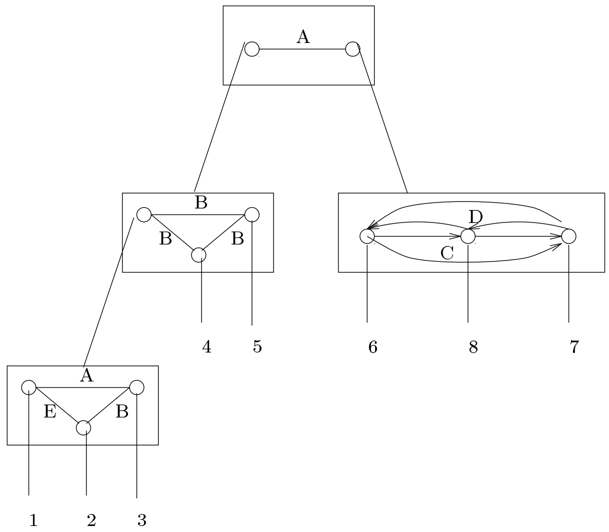
4. Let g be the ℓ2s such that the following ℓ2s-labeled tree family β is in $\mathbf{L2STF}$(g).



Give the shape of g without computing g, but rather by transforming β. Describe the transformations you apply and prove that the result is indeed the shape of g. **Answer:** the final answer is given in the figure below.



This is how to arrive at the answer. There are two criteria for a tree as given above to be a shape. First of all, all nodes should be labeled by special $\ell 2$s's; this is already the case in β. Secondly if a node is complete and one of its descendants is complete as well, then the label alphabet should be disjoint and analogously for a linear node. This means that $\{1, 2, 3\}$ is NOT to be moved into its parents, because this substructure is primitive. (If you would do this, then the result would not be special, so you can see for yourself that this is not correct.) There are two problems with disjointess: $\{4, 5\}$ with $\{1, 2, 3, 4, 5\}$ and $\{7, 8\}$ with $\{6, 7, 8\}$. In both cases we use the <u>sub</u> function defined in the reader to substitute the descendant into its parent. This makes sure that the resulting tree still represents g and examining the result tells us that the disjointness condition holds and that all node labels are special.

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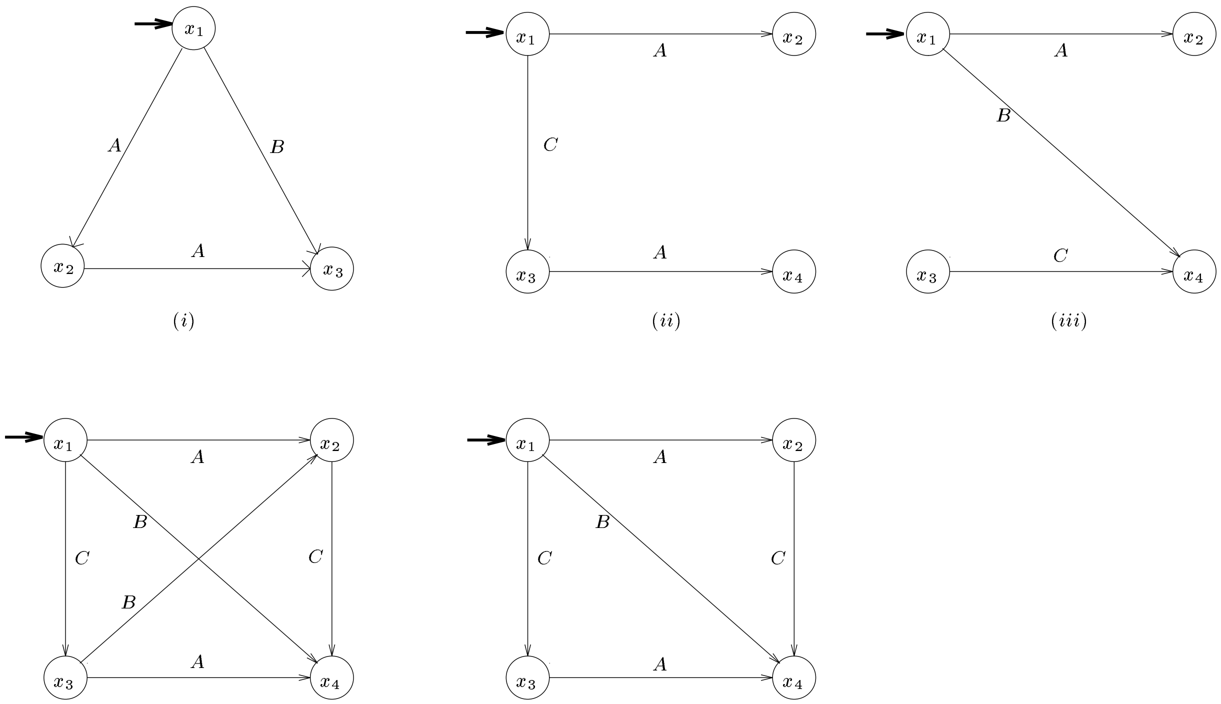
5. Which of the iℓp2s's given below is an abstract version of a sequential case graph? Prove each of your answers.



(i) (ii) (iii) (iv) (v)

Answer:

(i) Many people see directly that (i) contains three nodes with edges between them having the same label (in this case the label A). In the reader it is explained that this is impossible in an iℓp2s. If you don't see it, then you'll find that (i) is not NS: the only regions are the trivial ones.

(ii) Also not an iℓp2s. The problem here is that FC(regv((ii))) does not hold. To prove this we first must find the regional version of (ii). The regions are $\emptyset, D, Z_1 = \{x_1, x_2\}, Z = \{x_1, x_3\}$ and the complements $\overline{Z_1}$ and $\overline{Z_2}$. Then we find that in the regional version the edge C becomes the label $(\{Z_1\}, \{\overline{Z_1}\})$. Because the input side is also present in $R_{(ii)}(x_2)$ there should be an edge with this label leaving that node (arriving in $R_{(ii)}(x_4)$).

(iii) x_3 is not reachable from x_1, so (iii) is not an iℓp2s.

(iv) (iv) is not NS. The regions are $\emptyset, D, \{x_1, x_3\}$ and the complement of the latter. This means that the set of regions for x_1 and x_3 are equal (and of x_2 and x_4 as well).

(v) This is the only iℓp2s: all nodes are reachable from x_1, NS((v)) and ES((v)) hold and the regional version of (v) has the FC property.

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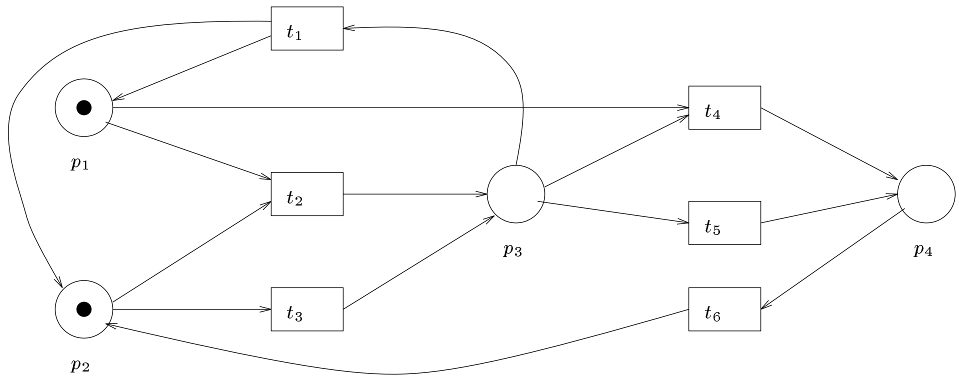
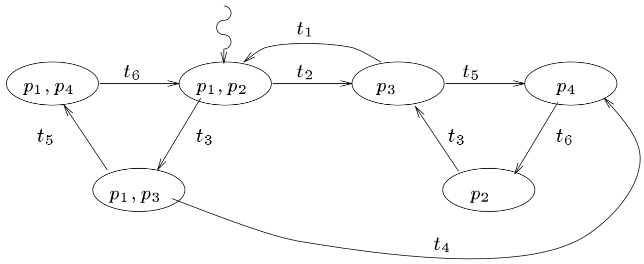
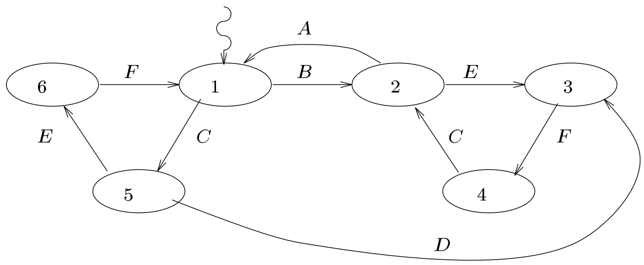
6. (i) Let N be the following EN system.



a. Give SCG(N).
Answer:



b. Give the regional version of an abstract version of SCG(N). (Hint: there are 10 regions.)
Answer: the ℓp2s g given below is an abstract version.



The regions are $\emptyset$, $Z_1 = \{1, 4\}$, $Z_2 = \{2, 5\}$, $Z_3 = \{3, 6\}$, $Z_4 = \{3, 5, 6\}$ and the five complements. Note that all non trivial regions are either $\underline{\text{sdom}}_{p_i}$ or a complement of such a set. Because the number of regions is given, you know these are all of them and need not look further. The regional version is found by replacing a node i by the set of regions of which i is part, and making the label of an edge (x, y) equal to the ordered symmetrical difference of x and y. So the label A stands for $(\{\overline{Z_1}, Z_2, \overline{Z_4}\}, \{Z_1, \overline{Z_2}, Z_4\})$.

(ii) Describe in your own words in at most 15 lines how to construct the "saturated" version of an EN system given the regional version of an abstract version of its sequential case graph.
Answer: let $g' = \underline{\mathrm{regv}}(g)$, with g the abstract version of $\mathrm{SCG}(N)$ and let $v \in \underline{\mathrm{dom}}(g')$ be the initial element. The saturated version of N is the EN system such that

- Every non trivial region of g is a place.
- Every label of g' is a transition.
- There is an arrow (p, t) for place p and transition t iff $p \in \underline{\mathrm{inp}}(t)$ (i.e., the region p is in the input set of a label t of the regional version).
- There is an arrow (t, p) for place p and transition t iff $p \in \underline{\mathrm{outp}}(t)$.
- The initial marking consists of all places/regions that are in v'.

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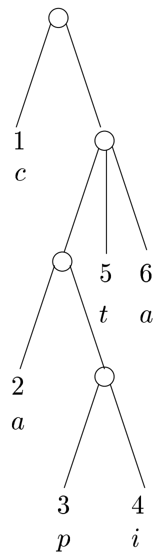
7. Let t be the following leaf-labeled ordered tree.



How many different texts are compatible with t? Give all of them.
Answer: first of all there are two texts compatible with t, the odd text and the even text. The latter is found by constructing a shape from the tree using the label VH in the root, the former by using $V\overline{H}$. Having done this we find the first linear order by reading the front of this shape. The second is found by reading the front of the shape, having inverted the order of the children of nodes containing the $V\overline{H}$ label. The following texts $\tau = (\lambda, \rho, \mu)$ and $\tau' = (\lambda, \rho, \mu')$ are the result, with λ such that $\lambda(1) = c, \lambda(2) = \lambda(6) = a, \lambda(3) = p, \lambda(4) = i$ and

$\lambda(5) = t$, $\rho = (1,2,3,4,5,6)$, $\mu = (1,6,5,2,4,3)$ and μ' is the reverse of μ being $(3,4,2,5,6,1)$.

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8. Let λ such that $\lambda(1) = \lambda(3) = \lambda(5) = a$ and $\lambda(2) = \lambda(4) = \lambda(6) = b$.

   (i) Let $\tau = (\lambda, \rho, \mu)$, with $\rho = (6,3,5,4,1,2)$ and $\mu = (2,1,4,3,5,6)$. Give the structured function of τ.
   Answer: the result is the ℓ2s g on a domain $\{1,2,3,4,5,6\}$ with the label of (i,j) equal to VH iff i comes before j in both orderings and $V\overline{H}$ iff i comes before j in ρ and after it in μ. Because the ℓ2s is reversible it is completely specified after stating that the reverse label of VH is $\overline{VH}$ and that of $V\overline{H}$ is $\overline{V}H$.

   (ii) Let $\tau_i = (\lambda, \rho_i, \mu_i)$ for $i = 1,2$, with $\rho_1 = \rho_2 = (1,2,3,4,5,6)$, $\mu_1 = (4,3,6,2,5,1)$, and $\mu_2 = (5,6,4,1,3,2)$.
   Which of these texts is alternating? Answer this *without* computing shape(τ_1) and shape(τ_2).
   Answer: this question should be answered by finding a primitive quartet or finding that there is no primitive quartet. In the first case, the text is not alternating, in the latter it is.
   It turns out that τ_1 is not alternating, because both $(2,4,5,6)$ and $(2,3,5,6)$ are primitive quartets. They are both suborderings of ρ and for instance in the first case, $(4,6,2,5)$ is a subordering of μ_1. Note that $(5,2,6,4)$ in μ_1 would have given us one as well. Also note that one primitive quartet is enough. You do not need two.
   The second text is alternating. Unfortunately this can only be proven by either finding the shape to be alternating or by checking all primitive quartets. It can also be seen more easily, by seeing that $\{1,2,3\}$ and $\{4,5,6\}$ are connected by the $V\overline{H}$ label and are thus disjoint clans. This means that we can never find a primitive structure on four nodes.

PUNTENVERDELING

Opgave	i	ii	iii	iv	v	totaal
1	6	6	-	-	-	12
2	4	6	-	-	-	10
3	6	3	6	-	-	15
4	-	-	-	-	-	8
5	3	3	3	3	3	15
6	14	4	-	-	-	18
7	-	-	-	-	-	10
8	5	7	-	-	-	12