Section 6

Context-Free and Non-Context-Free Languages

6 Context-Free and Non-Context-Free Languages
- Pumping Lemma
- Decision problems

Huiswerk. . .
Volgende week: geen werkcollege ’s middags?
Vragenuur: vrijdag 16 december, 13.15-15.00?

From lecture 2:
Regular language is language accepted by an FA.

Theorem

Suppose L is a language over the alphabet Σ. If L is accepted by a finite automaton M, and if n is the number of states of M, then

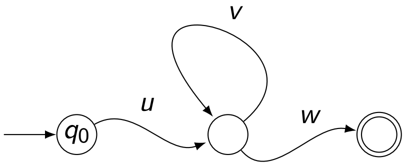
$\forall$ *for every $x \in L$*
satisfying $|x| \geq n$

$\exists$ *there are three string u, v, and w,*
such that $x = uvw$ and the following three conditions are true:
(1) $|uv| \leq n$,
(2) $|v| \geq 1$

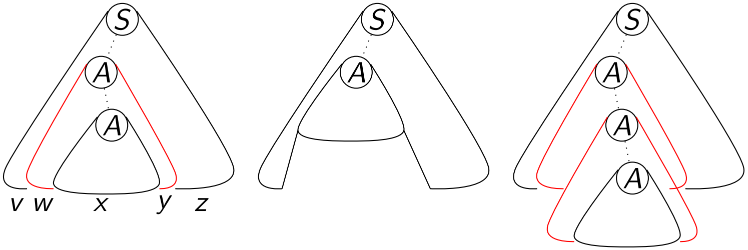
$\forall$ *and (3) for all $i \geq 0$, uv^iw belongs to L*

[M] Thm. 2.29

From lecture 2:



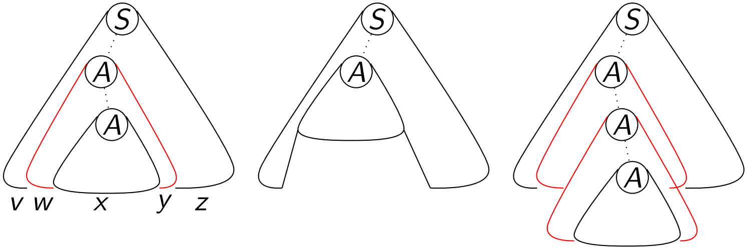
[M] Fig. 2.28


S
A
A
v w x y z
S
A
S
A
A
A

Pumping CF derivations

$$S \Rightarrow^* vAz \Rightarrow^* vwAyz \Rightarrow^* vwxyz, \; v, w, x, y, z \in \Sigma^*$$

$$S \underset{(1)}{\Rightarrow^*} vAz, \; A \underset{(2)}{\Rightarrow^*} wAy, \; A \underset{(3)}{\Rightarrow^*} x$$



$$S \underset{(1)}{\Rightarrow^*} vAz \underset{(3)}{\Rightarrow^*} vxz$$

$$S \underset{(1)}{\Rightarrow^*} vAz \underset{(2)}{\Rightarrow^*} vwAyz \underset{(2)}{\Rightarrow^*} vwwAyyz \underset{(3)}{\Rightarrow^*} vwwxyyz$$

Theorem (Pumping Lemma for context-free languages)

$\forall$ *for every context-free language* L

$\exists$ *there exists a constant* $n \geq 2$

such that

$\forall$ *for every* $u \in L$

with $|u| \geq n$

$\exists$ *there exists a decomposition* $u = vwxyz$

such that

(1) $|wy| \geq 1$

(2) $|wxy| \leq n$,

$\forall$ *(3) for all* $m \geq 0$, $vw^m xy^m z \in L$

[M] Thm. 6.1

Applying the Pumping Lemma

Example

AnBnCn is not context-free.

[M] E 6.3

Theorem (Pumping Lemma for context-free languages)

$\forall$ *for every context-free language* L

$\exists$ *there exists a constant* $n \geq 2$

such that

$\forall$ *for every* $u \in L$

with $|u| \geq n$

$\exists$ *there exists a decomposition* $u = vwxyz$

such that

(1) $|wy| \geq 1$

(2) $|wxy| \leq n$,

$\forall$ *(3) for all* $m \geq 0$, $vw^m xy^m z \in L$

If $L = L(G)$ with G in ChNF, then $n = 2^{|V|}$.

Proof. . .

[M] Thm. 6.1

From lecture 9:

Definition

CFG in *Chomsky normal form*

productions are of the form

– $A \to BC$ variables A, B, C

– $A \to \sigma$ variable A, terminal σ

Theorem

For every CFG G there is CFG G_1 in CNF such that $L(G_1) = L(G) - \{\Lambda\}$.

[M] Def 4.29, Thm 4.30

Theorem (Pumping Lemma for context-free languages)

[M] Thm. 6.1

Proof

Let G be CFG in Chomsky normal form with $L(G) = L - \{\Lambda\}$.

Derivation tree in G is binary tree
(where each parent of a leaf node has only one child).

Height of a tree is number of edges in longest path from root to leaf node.

At most 2^h leaf nodes in binary tree of height h: $|u| \leq 2^h$.

Theorem (Pumping Lemma for context-free languages)

[M] Thm. 6.1

Proof (continued)

At most 2^h leaf nodes in binary tree of height h: $|u| \leq 2^h$.

Let p be number of variables in G,
let $n = 2^p$
and let $u \in L(G)$ with $|u| \geq n$.

(Internal part of) derivation tree of u in G has height at least p.
Hence, longest path in (internal part of) tree contains at least $p + 1$ (internal) nodes.

Consider final portion of longest path in derivation tree.
(leaf node + $p + 1$ internal nodes),
with ≥ 2 occurrences of a variable A.

Pump up derivation tree, and hence u.

Application of pumping lemma:
mainly to prove that a language L cannot be generated by a context-free grammar.

How?
Find a string $u \in L$ with $|u| \geq n$ that cannot be pumped up!

What is n?
What should u be?
What can v, w, x, y and z be?
What should m be?

Suppose that there exists context-free grammar G with $L(G) = L$.
Let $n \geq 2$ be the integer from the pumping lemma.
We prove:
There exists $u \in L$ with $|u| \geq n$, such that
for every five strings v, w, x, y and z such that $u = vwxyz$
if
1. $|wy| \geq 1$
2. $|wxy| \leq n$

then
3. there exists $m \geq 0$, such that vw^mxy^mz does not belong to L

Example

AnBnCn is not context-free.

[M] E 6.3

$u = a^n b^n c^n$

$\{ x \in \{a, b, c\}^* \mid n_a(x) = n_b(x) = n_c(x) \}$

Example

XX is not context-free.

[M] E 6.4

Example

AnBnCn is not context-free.

[M] E 6.3

$u = a^n b^n c^n$

$\{\ x \in \{a, b, c\}^* \mid n_a(x) = n_b(x) = n_c(x)\ \}$

Example

XX is not context-free.

[M] E 6.4

$u = a^n b^n a^n b^n$

$\{\ a^i b^j a^i b^j \mid i, j \geq 0\ \}$

Example

$\{\ x \in \{a, b, c\}^* \mid n_a(x) < n_b(x)$ and $n_a(x) < n_c(x)\ \}$ is not context-free.

[M] E 6.5

ABOVE

$L = \{ x \in \{a, b, c\}^* \mid n_a(x) < n_b(x)$ and $n_a(x) < n_c(x) \}$ is not context-free.

Proof by contradiction.

Suppose L is context-free, then there exists a pumping constant n for L.

Choose $u = a^n b^{n+1} c^{n+1}$. Then $u \in L$, and $|u| \geq n$.

This means that we can pump u within the language L.

Consider a decomposition $u = vwxyz$ that satisfies the pumping lemma, in particular $|wxy| \leq n$.

Case 1: wy contains a letter a. Then wy cannot contain letter c (otherwise $|wxy| > n$). Now $u_2 = vw^2xy^2z$ contains more a's than u, so at least $n+1$, while u_2 still contains $n+1$ c's. Hence $u_2 \notin L$.

Case 2: wy contains no a. Then wy contains at least one b or one c (or both). Then $u_0 = vw^0xy^0z = vxz$ has still n a's, but less than $n+1$ b's or less than $n+1$ c's (depending on which letter is in wy). Hence $u_0 \notin L$.

These are two possibilities for the decomposition $vwxyz$, in both cases we see that pumping leads out of the language L.

Hence u cannot be pumped.

Contradiction; so L is not context-free.

Example

The Set of Legal C Programs is Not a CFL

[M] E 6.6

Choose $u =$

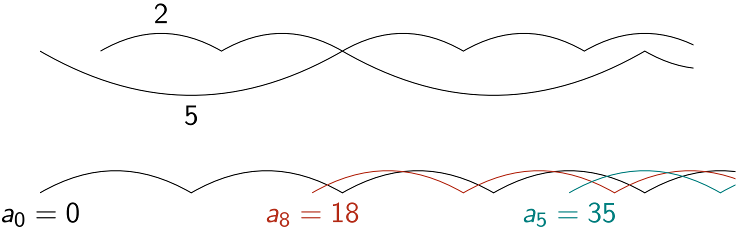
```
main(){int aaa...a;aaa...a=aaa...a;}
```

where aaa...a contains $n+1$ a's

Lemma (☒)

$L \subseteq \{a\}^*$ context-free, then L regular.

[M] Exercise 6.23



This exercise does not have to be known for the exam.

From lecture 3:

Prove the following generalization of the pumping lemma, which can sometimes make it unnecessary to break the proof into cases.

If L can be accepted by an FA,
then there is an integer n
such that for any $x \in L$ with $|x| \geq n$
and for any way of writing x as $x_1 x_2 x_3$ with $|x_2| = n$,
there are strings u, v and w such that

a. $x_2 = uvw$
b. $|v| \geq 1$
c. For every $m \geq 0$, $x_1 u v^m w x_3 \in L$

Generalization of pumping lemma for CFL:
pump at distinguished positions in u
Ogden's lemma does not have to be known for the exam.

From lecture 2:

FA $M_i = (Q_i, \Sigma, q_i, A_i, \delta_i) \quad i = 1, 2$

Product construction

construct FA $M = (Q, \Sigma, q_0, A, \delta)$ such that

– $Q = Q_1 \times Q_2$

– $q_0 = (q_1, q_2)$

– $\delta(\ (p, q), \sigma) = (\ \delta_1(p, \sigma), \delta_2(q, \sigma)\ )$

– A as needed

Theorem (2.15 Parallel simulation)

– $A = \{(p, q) \mid p \in A_1 \text{ or } q \in A_2\}$, *then* $L(M) = L(M_1) \cup L(M_2)$

– $A = \{(p, q) \mid p \in A_1 \text{ and } q \in A_2\}$, *then* $L(M) = L(M_1) \cap L(M_2)$

– $A = \{(p, q) \mid p \in A_1 \text{ and } q \notin A_2\}$, *then* $L(M) = L(M_1) - L(M_2)$

Proof. . .

[M] Sect 2.2

From lecture 6:

Regular languages are closed under

– Boolean operations (complement, union, intersection)
– Regular operations (union, concatenation, star)
– Reverse (mirror)
– [inverse] Homomorphism

From lecture 7:
Using building blocks

Theorem

If L_1, L_2 are CFL, then so are $L_1 \cup L_2$, $L_1 L_2$ and L_1^*.

$G_i = (V_i, \Sigma, S_i, P_i)$, having no variables in common.

Construction

$G = (V_1 \cup V_2 \cup \{S\}, \Sigma, S, P)$, new axiom S
– $P = P_1 \cup P_2 \cup \{S \to S_1, S \to S_2\}$ $L(G) = L(G_1) \cup L(G_2)$
– $P = P_1 \cup P_2 \cup \{S \to S_1 S_2\}$ $L(G) = L(G_1) L(G_2)$

$G = (V_1 \cup \{S\}, \Sigma, S, P)$, new axiom S
– $P = P_1 \cup \{S \to SS_1, S \to \Lambda\}$ $L(G) = L(G_1)^*$

[M] Thm 4.9

How about

- $L_1 \cap L_2$
- $L_1 - L_2$
- L_1'

for CFLs L_1 and L_2 ?

From lecture 8:

Example

AnBnCn is intersection of two context-free languages.

$L_1 = \{a^i b^i c^k \mid i, k \geq 0\}$
$L_2 = \{a^i b^k c^k \mid i, k \geq 0\}$
[M] E 6.10
Hence, CFL is not closed under intersection

Example

AnBnCn is intersection of two context-free languages.

[M] E 6.10

Hence, CFL is not closed under intersection

$L_1 \cap L_2 = (L_1' \cup L_2')'$

Hence, CFL is not closed under complement

$L_1' = \Sigma^* - L_1$

Hence, CFL is not closed under setminus

Example

Complement of XX
$= \{ x \in \{a,b\}^* \mid |x| \text{ is odd } \} \cup \{ x\,y \mid x, y \in \{a,b\}^*, |x| = |y|, x \neq y \}$
is context-free

[M] E 6.11

Indeed, CFL is not closed under complement