Eigenspaces of the Laplace-Beltrami-operator on $SL(n,\mathbb{R})/S(GL(1)\times GL(n-1))$. Part II

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5. THE POISSON-TRANSFORM

In this chapter we shall define the Poisson-transform $\mathscr{P}_s$ from $B(G/P;s)$ into $B(G/H;M_s)$. The main theorem of this paper states that for generic s, $\mathscr{P}_s$ is an isomorphism between these two spaces. This will be shown by exhibiting an explicit inverse, namely the boundary value map.

Let dg be a Haarmeasure on G, and dx a G-invariant measure on X. Using these measures, continuous functions can be considered as distributions. Furthermore, right H-invariant distributions on G are identified with distributions on X. Let $i\in\{1,2\}$.

LEMMA 5.1. *The Poissonkernels P_s^1 and P_s^2, as defined in Chapter 2, are for* Re $s<n-1$ *locally integrable on G and on X. They are meromorphically extended to $\mathbb{C}$ as distributions on G and X. The set of poles is contained in $\mathbb{Z}$. The extensions satisfy (M_s) on X, and their behaviour under left translation by elements of $\bar{P}$ is the same as that of the original functions.*

PROOF. Choosing suitable coordinates on X, one is led to integrals like

$$\int_{\mathbb{R}^2} |xy|^{\lambda} f(x,y)d(x,y) \text{ and } \int_{\mathbb{R}} |x|^{\lambda} f(x)dx$$

for $\lambda\in\mathbb{C}$ and C_c^∞-functions f (Cf. [Kosters, 9]). These integrals can be handled as in [Gel'fand, Shilov, 4]. Furthermore, for Re s sufficiently small, P_s^i satisfies (M_s). This follows from Corollary 2.3.3 in [Kosters, 9], or from direct computation using Lemma 4.1. Uniqueness of analytic continuation yields the desired properties. The details are left to the reader. ■

Now we want to give a meaning to the integral

$$\int_K f(k)P_s^i(k^{-1}g)dk \quad (g\in G)$$

for hyperfunctions f on K. In order to describe hyperfunctions on X, let $\{\Omega_l\}_{l\in L}$ be an open cover of X such that every Ω_l has compact closure, which is contained in some coordinate patch on X. A hyperfunction h on X is given by elements

$$h_l\in A'(\bar{\Omega}_l)/A'(\partial\Omega_l)=B(\Omega_l).$$

Here the primes denote topological duals, in the well-known sense. Now fix $l\in L$. Take $F\in A(\bar{\Omega}_l)$, so F is a real analytic function on some open neighbourhood of $\bar{\Omega}_l$. Let for $k\in K$ and $x\in X$:

$$(l_kF)(x)=F(kx).$$

Then $l_kF\in A(k^{-1}\bar{\Omega}_l)$. Consider the integral

$$\int_{k^{-1}\bar{\Omega}_l} P_s^i(x)(l_kF)(x)dx.$$

Using coordinates as in the proof of Lemma 5.1., this integral can be meromorphically extended to $\mathbb{C}$. Indeed, the problem reduces to the continuation of integrals like

$$\int_\Omega |xy|^\lambda f(x,y)d(x,y) \quad (\lambda\in\mathbb{C},\ f\in A(\Omega))$$

for suitable compact subsets Ω of $\mathbb{R}^2$. Note that the expression analytically depends on k. Using the fact that $B(K)$ is the dual of $A(K)$, $\mathscr{P}_s^if$ can be defined by

$$\langle \mathscr{P}_s^if,F\rangle=\langle f(k),\ \int_{k^{-1}\bar{\Omega}_l} P_s^i(x)(l_kF)(x)dx\rangle$$

for $f\in B(K)$. This expression continuously depends on F.

Now take the class of $\mathscr{P}_s^if$ in $B(\Omega_l)$ and remark that on intersections $\Omega_l\cap\Omega_{l'}$ the definitions coincide. Therefore, we see that $\mathscr{P}_s^if\in B(X)$ for $f\in B(K)$.

If f happens to be in $C(K)$, then $\mathscr{P}_s^if$ is a function on X:

$$(\mathscr{P}_s^if)(gH)=\int_K P_s^i(k^{-1}g)f(k)dk \quad (g\in G).$$

This remark uses the embedding $C(K)\to B(K)$, given by

$$\langle f,\psi\rangle=\int_K f(k)\psi(k)dk \quad (f\in C(K),\ \psi\in A(K)).$$

The next step is to show that $\mathscr{P}_s^if$ satisfies (M_s) on $\mathbb{X}^+$. Therefore, we need to describe the action of $\Box$ on $B(X)$. Take $\{\Omega_l\}_{l\in L}$ as before and define for $h\in B(X)$

$$\langle \Box h,F\rangle=\langle h,\Box F\rangle \quad (F\in A(\bar{\Omega}_l)).$$

Note that for C^2-functions this definition is compatible with the usual one. Indeed, if $h \in C(X)$, then

$$\langle h, F\rangle = \int_{\bar{\Omega}_l} h(x)F(x)dx$$

defines a hyperfunction on X. Now use the fact that $\Box$ is self-adjoint with respect to dx, and note that all boundary value terms that occur are zero in $B(\Omega_l)$.

An easy application of Lemma 5.1 shows that

$$\Box \mathscr{P}_s^i f = (s^2 - \varrho^2) f$$

whence $\mathscr{P}_s^i f \in B(G/H; M_s)$. Now we can define the Poisson-transform $\mathscr{P}_s f$ of an element f of $B(G/P; s)$. Therefore, write $f = f_1 + f_2$ with $f_j \in B_j(G/P; s)$ $(j = 1, 2)$, restrict f_j to K, and define

$$\mathscr{P}_s f = \mathscr{P}_s^1 f_1 + \mathscr{P}_s^2 f_2.$$

Combining all these remarks we get:

LEMMA 5.2. *Let $s \in \mathbb{C}$, $s \notin \mathbb{Z}$. Then $\mathscr{P}_s$ maps $B(G/P; s)$ into $B(G/H; M_s)$.* ■

LEMMA 5.3. *Let $s \in \mathbb{C}$, $s \notin \mathbb{Z}$, $f \in B_i(G/P; s)$, $g \in G$. Then $\mathscr{P}_s^i(\tilde{\pi}_s(g)f) = \pi_s(g)\mathscr{P}_s^i f$: $\mathscr{P}_s^i$ is G-equivariant.*

PROOF. We only consider $i = 1$. If $g \in G$, define $\kappa(g) \in K/K \cap \bar{M}$ and $t(g) \in \mathbb{R}$ by $g \in \kappa(g)\bar{M}a_{t(g)}N$ (Cf. Theorem 2.9.). For $\phi \in C(K/K \cap \bar{M})$ we have

$$\int_K \phi(k)dk = \int_K \phi(\kappa(gk))e^{-2\varrho t(gk)}dk \quad (g \in G)$$

([Varadarajan, 16], p. 294). By transposition, this formula is also valid for $\phi \in B(K/K \cap \bar{M})$. Let Ω_l be as before. We get

$$\begin{aligned}
&\int_K f(k) \int_{k^{-1}g\bar{\Omega}_l} P_s^i(x)(l_k l_{g^{-1}} F)(x)dxdk = \\
&= \int_K f(\kappa(gk)) \int_{\kappa(gk)^{-1}g\bar{\Omega}_l} P_s^i(x)(l_{g^{-1}\kappa(gk)}F)(x)dx \quad e^{-2\varrho t(gk)}dk = \\
&= \int_K f(gk) \int_{ma_{t(gk)}n\bar{\Omega}_l} P_s^i(x)(l_{n^{-1}a_{-t(gk)}m^{-1}} l_k F)(x)dx \quad e^{(-s-\varrho)t(gk)}dk = \\
&= \int_K f(gk) \int_{k^{-1}\bar{\Omega}_l} P_s^i(x)(l_k F)(x)dxdk
\end{aligned}$$

where $gk = \kappa(gk)ma_{t(gk)}n$ with $m \in \bar{M}$, $n \in N$. From these equations the lemma easily follows. ■

Now we can state the main theorem of this paper:

THEOREM 5.4. *Let $s \in \mathbb{C}$, $2s \notin \mathbb{Z}$. Then $\mathscr{P}_s$ is a G-equivariant isomorphism of $B(G/P; s)$ onto $B(G/H; M_s)$.*

The proof of this theorem is based on two lemmas:

LEMMA 5.5. *Let $s \in \mathbb{C}$, $2s \notin \mathbb{Z}$, $i \in \{1,2\}$. Then there exist nonzero complex numbers $c_i(s)$ such that for all $f \in B_i(G/P;s) : \beta_s \mathscr{P}_s^i f = c_i(s) f$.*

PROOF. This is given in Chapter 6. There the $c_i(s)$ are explicitly computed. ■

LEMMA 5.6. *Let $s \in \mathbb{C}$, $2s \notin \mathbb{Z}$. Then β_s is injective.*

PROOF. This is given in Chapter 7. ■

Now we can give the

PROOF OF THEOREM 5.4. We only need to show that $\mathscr{P}_s$ is injective and surjective.

i) $\mathscr{P}_s$ is injective. Suppose that $\mathscr{P}_s f = 0$ for some $f \in B(G/P;s)$. Then Lemma 5.5 implies that

$$0 = \beta_s \mathscr{P}_s f = \beta_s(\mathscr{P}_s^1 f_1 + \mathscr{P}_s^2 f_2) = c_1(s) f_1 + c_2(s) f_2$$

whence $f_1 = f_2 = 0$.

ii) $\mathscr{P}_s$ is surjective. Let $u \in B(G/H;M_s)$. Define

$$u' = \frac{1}{c_1(s)} \mathscr{P}_s^1(\beta_s u)_1 + \frac{1}{c_2(s)} \mathscr{P}_s^2(\beta_s u)_2.$$

Then $u' \in B(G/H;M_s)$, and $\beta_s(u-u') = 0$, which follows from Lemma 5.5. Applying Lemma 5.6 we see that $u - u' = 0$, thereby showing that

$$u = \mathscr{P}_s\left(\frac{1}{c_1(s)} (\beta_s u)_1 + \frac{1}{c_2(s)} (\beta_s u)_2\right).$$ ■

6. PROOF OF LEMMA 5.5

From now on $i \in \{1,2\}$, $s \in \mathbb{C}$, $2s \notin \mathbb{Z}$. In this chapter we compute $\beta_s P_s^i$, using the theory of Bruhat as it is presented in [Kashiwara et al., 7], Appendix. Note that for $j = 1,2$:

$$(\beta_s P_s^i)_j \in B_j(G/P;s).$$

In order to compute $\beta_s P_s^i$, we derive some properties. The G-equivariance of β_s implies that

$$(\beta_s P_s^1)_j(m a_t n g) = e^{(s+\varrho)t}(\beta_s P_s^1)_j(g)$$

$$(\beta_s P_s^2)_j(m a_t n g) = \chi(m) e^{(s+\varrho)t}(\beta_s P_s^2)_j(g)$$

for $m \in \bar{M}$, $t \in \mathbb{R}$, $n \in N$ and $g \in G$. It appears that these properties determine $\beta_s P_s^i$ uniquely.

Consider the following distributions on G:

$$\langle \delta_s^1, f\rangle = \int_{\bar{M}} \int_{\mathbb{R}} \int_{N} f(ma_t n) e^{(s+\varrho)t} dm dt dn$$

$$\langle \delta_s^2, f\rangle = \int_{\bar{M}} \int_{\mathbb{R}} \int_{N} f(ma_t n) \chi(m) e^{(s+\varrho)t} dm dt dn$$

for $f \in D(G)$; here dm and dn are Haarmeasures on $\bar{M}$ and N, resp., and dt is the Lebesguemeasure on $\mathbb{R}$. Using the commutation rules for elements of $\bar{M}$, A and N, derived in Chapter 2, it is easily seen that δ_s^i has the same properties as $(\beta_s P_s^i)_i$. We shall prove:

LEMMA 6.1. *$\beta_s P_s^i = c_i(s)\delta_s^i$ for certain complex numbers $c_i(s)$.*

The proof of this lemma requires some preparations. Using the techniques of [Faraut, 2], it is easily proved that

$$\langle \delta_s^1|_K, f\rangle = \int_{K\cap\bar{M}} f(k\dot{m}) d\dot{m}$$

$$\langle \delta_s^2|_K, f\rangle = \int_{K\cap\bar{M}} f(k\dot{m}) \chi(\dot{m}) d\dot{m}$$

for $f \in D(K)$, where $d\dot{m}$ is the normalized Haarmeasure on $K \cap \bar{M}$.

Now we give the Bruhat-decomposition of G with respect to $\bar{P}$: we want to describe the structure of $\bar{P}\backslash G/\bar{P}$. The Weylgroup W is by definition the quotient $M^*_{\min}/M_{\min}$, where $M^*_{\min}$ is the normalizer of $A_{\min}$ in K. It is well-known that W is isomorphic to S_n, the group of permutations of $\{1, 2, \ldots, n\}$. In the diagonal-form, W can be realized in G as the set of matrices with exactly one 1 in every row and column, and eventually a minus sign in the last column to insure determinant one. Define:

$$w_1 = \begin{pmatrix} 1 & & \Theta \\ & \ddots & \\ \Theta & & 1 \end{pmatrix},$$

$$w_2 = \begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 & -1 \\ \cdot & 1 & & & \Theta & 0 \\ \cdot & & \ddots & & & \vdots \\ 0 & \Theta & & & 1 & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & \Theta & \\ & & 1 & & & \\ & & & \ddots & & \\ & & \Theta & & 1 & \\ & & & & & -1 \end{pmatrix},$$

$$w_4 = \begin{pmatrix} 1 & & & & \\ & \ddots & & \Theta & \\ & & 1 & & \\ & \Theta & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}, \quad w_5 = \begin{pmatrix} 0 & 0 & 0 & & & -1 \\ 1 & 0 & 0 & & & 0 \\ 0 & 0 & 1 & & \Theta & \vdots \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \Theta & & 1 & 0 \\ 0 & 1 & & & 0 & 0 \end{pmatrix},$$

$$w_6 = \begin{pmatrix} 0 & 1 & 0 & & & 0 \\ 0 & 0 & 0 & & & -1 \\ \cdot & \cdot & 1 & \Theta & & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & \Theta & & 1 & 0 \\ 1 & 0 & & & 0 & 0 \end{pmatrix}, \quad w_7 = \begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & & & \Theta & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & & \Theta & & & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix}$$

Of course, w_7 is only defined for $n>3$. We have:

LEMMA 6.2. 1) *$G=\bigcup_{i=1}^{7} \bar{P}'w_i\bar{P}'$, a disjoint union* $(n>3)$

2) *$G=\bigcup_{i=1}^{6} \bar{P}'w_i\bar{P}'$, a disjoint union* $(n=3)$.

Here primes denote the diagonal-form.

PROOF. [Warner, 18], 1.2.4. shows that there is a one-to-one correspondence between $\bar{P}\setminus G/\bar{P}$ and $S_n'\setminus S_n/S_n'$, where $S_n'=\{\sigma\in S_n|\sigma(1)=1,\ \sigma(n)=n\}$. For $n=3$, $S_n'=\{1\}$. For $n>3$, one easily finds the representatives $\{w_1,\ldots,w_7\}$ for $S_n'\setminus S_n/S_n'$.

It is also possible to proceed more directly, without the use of the general theory from [Warner, 18]. Then we use the Bruhat-decomposition with respect to $P_{\min}$ and arguments like those in the proof of Lemma 2.4. ∎

Notice, that this proves Theorem 2.8. 1).

LEMMA 6.3. *Let $u\in B(G)$. Suppose that for all $t,x\in\mathbb{R}$, $n,n'\in N$, $g\in G$:*

$$u(a_t n g a_x n') = e^{(s+\varrho)t+(s-\varrho)x}u(g)$$

1) *If $u(mgm')=u(g)$ for all $m,m'\in\bar{M}$, $g\in G$, then $u=c\delta_s^1$ for some $c\in\mathbb{C}$.*

2) *If $u(mgm')=\chi(mm')u(g)$ for all $m,m'\in\bar{M}$, $g\in G$, then $u=c\delta_s^2$ for some $c\in\mathbb{C}$.*

3) *If $u(mgm')=\chi(m)u(g)$ for all $m,m'\in\bar{M}$, $g\in G$, then $u=0$.*

4) *If $u(mgm')=\chi(m')u(g)$ for all $m,m'\in\bar{M}$, $g\in G$, then $u=0$.*

PROOF. The proof of this lemma is a copy of the proof of the proposition on p. 33 in [Kahiwara et al., 7]. Let us consider case 1). Define

$$H_1 = \begin{pmatrix} 1 & & \\ & \Theta & \\ & & -1 \end{pmatrix} \in \mathfrak{a}'$$

and

$$\chi_1(ma_t n, m_1 a_x n_1) = e^{(s+\varrho)t-(s-\varrho)x} \quad (m,m_1\in\bar{M}'; t,x\in\mathbb{R}, n,n_1\in N').$$

Here the primes refer to the diagonal-form. χ_1 is a character of the group $\bar{P}'\times\bar{P}'$. Note that H_1 acts on

$$\mathfrak{g}_w = \mathfrak{g}/(\mathfrak{m}'+\mathfrak{a}'+\mathfrak{n}'+\mathrm{Ad}\,(w^{-1})(\mathfrak{m}'+\mathfrak{a}'+\mathfrak{n}'))$$

for $w \in W$. In particular, we are interested in the eigenvalues of ad H_1 on $\mathfrak{g}_w$ and in the numbers $d\chi_1(\mathrm{Ad}\,(w)H_1, H_1)$, for $w \in \{w_1, \ldots, w_7\}$. All these facts are given in Table 6.I. Instead of giving the whole proof, the reader is asked to fit in the details. Here, we shall sketch the main ideas. In [Kashiwara et al., 7] a sufficient condition is given for the (non-)existence of hyperfunctions on a manifold, which have certain transformation properties under the action of a Lie group. This condition is an integrality condition on certain eigenvalues. An application in our case yields that a hyperfunction u on G, with the properties mentioned above, has its support in $\bar{P}'$, and is unique up to a complex constant, under the condition that $s \notin \mathbb{Z}$. In fact, one uses the previous lemma and a case by case examination of the subsets $\bar{P}'w_i\bar{P}'$ $(i = 1, \ldots, 7)$. For example, for w_7 we get the integrality condition $s \notin \{n-4, n-5, \ldots\}$. For w_5 and w_6 we get the condition $3s \notin \{n-3, n-5, \ldots\}$. In order to handle the case $3s \in \mathbb{Z}$, $s \notin \mathbb{Z}$, H_1 can be replaced by

$$H_2 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \text{\Large 0} \\ & & 0 & & \\ & & & \ddots & \\ & & & 0 & \\ & \text{\Large 0} & & & -2 \end{pmatrix} \in \mathfrak{a}'_{\mathfrak{p}}.$$

Then the condition becomes $s \notin \{-1, -2, -3, \ldots\}$. The only set that possibly carries a hyperfunction with the properties mentioned above is $\bar{P}'$; furthermore there is — up to a scalar — at most one such hyperfunction. The fact that δ_s^1 satisfies the same properties as u, and is nonzero, now gives the lemma. The proof even works for $2s \in \mathbb{Z}$, $s \notin \mathbb{Z}$.

Case 2) is handled in a similar way. Now let us consider case 3).

Arguing as before, we come to the conclusion that u is a distribution on G, with support contained in $\bar{P}'$, of the form

$$\langle u, f \rangle = \int_{\bar{M}} \int_{\mathbb{R}} \int_{N} f(ma_t n) H(m, t, n)\, dm\, dt\, dn \quad (f \in D(G))$$

where H is a real analytic function on $\bar{M} \times \mathbb{R} \times N$. The fact that u is a distribution also comes from the theory of Kashiwara et al. The right N-invariance shows that $H(m, t, n)$ does not depend on n. It is easy to prove that $H(mm_1m_2, t, e) = \chi(m)H(m_1, t, e)$ for all $m, m_1, m_2 \in \bar{M}$, $t \in \mathbb{R}$. Some special choices for m, m_1 and m_2, using the element

$$\begin{pmatrix} 1 & & & & & \\ & -1 & & \text{\Large 0} & & \\ & & 1 & & & \\ & & & \ddots & & \\ & \text{\Large 0} & & & 1 & \\ & & & & & -1 \end{pmatrix}$$

of $\bar{M}$, easily imply that $H(m,t,n)=0$ for all $m\in\bar{M}$, $t\in\mathbb{R}$, $n\in N$. Whence $u=0$. Case 4) is handled in the same way. We finish the proof by giving the table which was used in case 1).

Table 6.1.

w	$\mathfrak{g}_w$	eigenvalues of ad H_1 on $\mathfrak{g}_w$	$d\chi_1(\mathrm{Ad}(w)H_{11}H_1)$
w_1	$\left\{\begin{pmatrix} 0 & & & \\ * & & \theta & \\ \vdots & & & \\ * & \cdots & * & 0 \end{pmatrix}\right\}$	-1 ($2n-4$ times) -2 (once)	2ϱ
w_2	$\{(\theta)\}$	none	$-2s$
w_3	$\left\{\begin{pmatrix} & \theta & & \\ * & \cdots & * & 0 \end{pmatrix}\right\}$	-1 ($n-2$ times) -2 (once)	$\dfrac{-s+3\varrho}{2}$
w_4	$\left\{\begin{pmatrix} 0 & \\ * & \\ \vdots & \theta \\ * & \end{pmatrix}\right\}$	-1 ($n-2$ times) -2 (once)	$\dfrac{-s+3\varrho}{2}$
w_5	$\left\{\begin{pmatrix} & & \theta & & \\ 0 & * & 0 & \cdots & 0 \end{pmatrix}\right\}$	-1 once	$\dfrac{-3s+\varrho}{2}$
w_6	$\left\{\begin{pmatrix} 0 & \\ * & \\ 0 & \theta \\ \vdots & \\ 0 & \end{pmatrix}\right\}$	-1 (once)	$\dfrac{-3s+\varrho}{2}$
w_7	$\left\{\begin{pmatrix} 0 & & & & \\ \vdots & & & & \\ 0 & & & \theta & \\ * & & & & \\ 0 & * & 0 & \cdots & 0 \end{pmatrix}\right\}$	-1 (twice)	$-s+\varrho$

■

Now it is easy to give the proof of Lemma 6.1, because this is a direct consequence of the previous lemma and the transformation properties of the hyperfunctions involved.

Take $f \in B_1(G/P;s)$. Then:

$$\beta_s(\mathscr{P}_s^1 f) = \beta_s(\int_K f(k) l_{k^{-1}} P_s^1 dk) = \int_K f(k) l_{k^{-1}}(\beta_s P_s^1) dk =$$

$$= \int_K f(k) l_{k^{-1}}(\beta_s P_s^1)_1 dk = \int_K f(k) c l_{k^{-1}} \delta_s^1 dk = cf$$

where $c = c_1(s) \in \mathbb{C}$, and $l_k P_s^1(x) = P_s^1(kx)$ $(x \in G/H, k \in K)$. Here we used the G-equivariance of β_s, Lemma 6.3, and the argument of [Kashiwara et al., 7], p. 22, which enables one to take β_s under the integral. So we have

$$\beta_s \mathscr{P}_s^1 f = c_1(s) f \quad (f \in B_1(G/P;s)).$$

In the same way it can be proved that:

$$\beta_s \mathscr{P}_s^2 f = c_2(s) f \quad (f \in B_2(G/P;s)).$$

In order to complete the proof, we have to show that $c_i(s) \neq 0$. Therefore these numbers are explicitly computed.

LEMMA 6.4. *If $2s \notin \mathbb{Z}$, then*

$$c_1(s) = \frac{(n-2)!}{\pi^2} 2^{2-s} \Gamma(s) \Gamma\left(\frac{-s-n+3}{2}\right)^2 \cos\frac{\pi s}{2} \cos^2\frac{\pi}{4}(-s-n+3)$$

$$c_2(s) = -tg^2 \frac{\pi}{4}(-s-n+3) \cdot c_1(s).$$

In particular $c_1(s) \neq 0$ and $c_2(s) \neq 0$.

PROOF. First we compute the Poisson-transform of the element of $B_1(G/P;s)$ which is equal to 1 on K; this element is unique and will be denoted by 1. We have:

$$(\mathscr{P}_s^1 1)(g) = \int_K P_s^1(k^{-1} g) dk \quad (g \in G)$$

In [Kosters, 9], p. 106 ff, we find, with $k \in K$, $t \in \mathbb{R}$, $h \in H$:

$$(\mathscr{P}_s^1 1)(k a_t h) = \frac{2^{(-s-\varrho)/2} \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{-s-\varrho+2}{4}\right)^2}{\pi \Gamma\left(\frac{-s+1}{2}\right)} (\mathrm{ch}\ 2t)^{(-s-\varrho)/2}$$

$$ {}_2F_1\left(\frac{\varrho+s}{4}, \frac{\varrho+s}{4}; \frac{\varrho}{2}; \mathrm{th}^2\ 2t\right).$$

To prove this formula, note that $\mathscr{P}_s^1 1$ is left K- and right H-invariant, and use the fact that $\mathscr{P}_s^1 1$ satisfies the equation (M_s). This leads to a hypergeometric differential equation on A, which has a unique analytic solution. That $\mathscr{P}_s^1 1$ is analytic will be shown in the proof of Lemma 7.1. Moreover,

$$(\mathscr{P}_s^1 1)(e) = \int_K P_s^1(k) dk$$

can be computed directly. Note that $\mathcal{P}_{-s}^1 1 = d(s)\mathcal{P}_s^1 1$, for some $d(s) \in \mathbb{C}$. Since $\beta_s \mathcal{P}_s^1 1 = c_1(s) \cdot 1$ and $\beta_{-s}\mathcal{P}_s^1 1 = c_1'(s) \cdot 1$, for certain complex numbers $c_1(s)$ and $c_1'(s)$, $\mathcal{P}_s^1 1$ has analytic boundary values, and therefore $\mathcal{P}_s^1 1$ is a so-called ideally analytic solution of (M_s) (cf. [Oshima, Sekiguchi, 13], p. 25, 26 and 59). This implies that it is possible to compute $\beta_s(\mathcal{P}_s^1 1)(e)$ by taking a limit:

$$\beta_s(\mathcal{P}_s^1 1)(e) = \lim_{t \downarrow 0} t^{(s-\varrho)/2}(\mathcal{P}_s^1 1)(a(t)) =$$

$$= \frac{2^{-s}}{\pi} \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right) \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{-s+1}{2}\right)} \frac{\Gamma\left(\frac{-s-\varrho+2}{4}\right)^2}{\Gamma\left(\frac{s+\varrho}{4}\right)^2} \quad (\mathrm{Re}\, s > 0)$$

according to some well-known properties of the hypergeometric function (cf. [Sekiguchi, 15], p. 180 or [Erdélyi, 1], Chapter 2). Using the formula

$$\frac{\Gamma(z)}{\Gamma(-z+\frac{1}{2})} = \frac{\Gamma(2z)\cos \pi z}{\sqrt{\pi}} 2^{1-2z}$$

(cf. [Erdélyi, 1], Chapter 1) and the fact that

$$\beta_s(\mathcal{P}_s^1 1)(e) = c_1(s) 1(e) = c_1(s)$$

the first part of the lemma is easily proved. However, the limiting process used above is only justified for Re $s > 0$. Meromorphic continuation extends the result to $\mathbb{C} - \frac{1}{2}\mathbb{Z}$. Note that the meromorphic continuation of $c_1(s)$ has zeroes contained in $\{1, 3, 5, 7, \ldots\} \cup \{-n+1, -n-3, -n-7, \ldots\}$ and poles contained in $\{0, -2, -4, \ldots\} \cup \{-n+3, -n+7, -n+11, \ldots\}$, perhaps cancelling one another. In particular, $c_1(s) \neq 0$ if $2s \notin \mathbb{Z}$.

It is also possible to compute $c_1(s)$ using the methods of [Sekiguchi, 15], § 7. Now we compute $c_2(s)$. Therefore we consider the function Y in $B_2(G/P; s)$ defined by

$$Y(k) = 2(k_{1n}k_{n1} - k_{11}k_{nn}) \quad (k \in K)$$

This function plays an important role in [Kosters, 9] and will also be of great importance in Chapter 7. A computation, similar to that of $\mathcal{P}_s^1 1$, yields:

$$(\mathcal{P}_s^2 Y)(a_t h) = \frac{2^{(-s-\varrho)/2} 4\Gamma\left(\frac{n}{2}\right)}{\pi(n-1)} \frac{\Gamma\left(\frac{-s-\varrho+4}{4}\right)^2}{\Gamma\left(\frac{-s+1}{2}\right)} \mathrm{th}\, 2t\, (\mathrm{ch}\, 2t)^{(-s-\varrho)/2}$$

$${}_2F_1\left(\frac{\varrho+s+2}{4}, \frac{\varrho+s+2}{4}; \frac{\varrho+2}{2}; \mathrm{th}^2\, 2t\right)$$

for $t\in\mathbb{R}$, $h\in H$ (cf. [Kosters, 9], 4.6). Again, $\mathscr{P}_s^2 Y$ is an ideally analytic solution of (M_s), so we can take a suitable limit, and derive:

$$-2c_2(s)=\lim_{t\downarrow 0} t^{(s-\varrho)/2}(\mathscr{P}_s^2 Y)(a(t))=$$

$$=\frac{(n-2)!}{\pi^2}2^{3-s}\Gamma(s)\Gamma\left(\frac{-s-n+3}{2}\right)\cos\frac{\pi s}{2}\cos^2\frac{\pi}{4}(-s-n+1)\ (\mathrm{Re}\,s>0)$$

from which the lemma easily follows. ■

7. PROOF OF LEMMA 5.6

In this chapter we prove that for complex s with $2s\notin\mathbb{Z}$, β_s is injective. Note that it is sufficient to show that if $u\in B(G/H;M_s)$ satisfies $\beta_s u=0$, then $\beta_{-s}u=0$. This is a consequence of Proposition 2.15 in [Oshima, Sekiguchi, 13], where it is proved that $\mathbb{X}^0$ has an open neighbourhood V in $\mathbb{X}$ with the following property: if $u\in B(G/H;M_s)$ satisfies $\beta_s u=\beta_{-s}u=0$, then $u|_{V\cap\mathbb{X}^+}=0$. Now the G-equivariance of β_s implies Lemma 5.6. Throughout this chapter $s\in\mathbb{C}$, $2s\notin\mathbb{Z}$.

Let us give a brief outline of the contents of this chapter. Take $u\in B(G/H;M_s)$ with $\beta_s u=0$. Then Lemma 7.1 shows that $(\beta_{-s}u)_1=0$. However, it is more difficult to prove that $(\beta_{-s}u)_2=0$. Therefore, we consider a certain representation of G and some of its matrix coefficients. These are used to construct a cyclic vector for a certain principal series representation of G. The argument is completed by some careful integral manipulations.

LEMMA 7.1. *Let $u\in B(G/H;M_s)$. If $\beta_s u=0$, then $(\beta_{-s}u)_1=0$.*

PROOF. (Cf. [Sekiguchi, 15], Lemma 8.3.)

Take an arbitrary u in $B(G/H;M_s)$; $g\in G$. Define the hyperfunction V_g by

$$V_g(g')=\int_K u(gkg')dk\quad(g'\in G).$$

Then V_g is a hyperfunction on X and satisfies (M_s), because $u\in B(G/H;M_s)$ and $\Box$ is G-invariant. Note that V_g is left K-invariant and hence real analytic, because it can be considered as an eigenfunction of the Casimir-operator on $K\setminus G$, which is elliptic. It is easy to see that $\mathscr{P}_s^1 1|_A$ and $V_g|_A$ satisfy the same differential equation (see p. 106 in [Kosters, 9]), which has only one real analytic solution. The fact that $\mathscr{P}_s^1 1|_A$ is a nonzero solution shows that there is a complex number $\upsilon(g)$ such that

$$V_g=\upsilon(g)\mathscr{P}_s^1 1.$$

Using Lemma 5.5 one derives

$$\beta_s V_g=\upsilon(g)c_1(s)1.$$

Furthermore, following [Schlichtkrull, 14], p. 75, it can be seen that

$$\beta_s\left(\int_K u(gk\cdot)dk\right)=\int_K\beta_s u(gk\cdot)dk$$

and it follows that

$$v(g)c_1(s)=\beta_s V_g(e)=\int_K (\beta_s u)(gk)dk.$$

Now suppose that $\beta_s u=0$. The fact that $c_1(s)\neq 0$ implies that $v(g)=0$, whence $V_g=0$. Thus

$$\int_K (\beta_{-s}u)(gk)dk=0 \quad (g\in G).$$

Because of the transformation properties under right translation by w, it is easily seen that

$$\int_K (\beta_{-s}u)_2(gk)dk=0 \quad (g\in G)$$

and therefore

$$\int_K (\beta_{-s}u)_1(gk)dk=0$$

for all $g\in G$. By Lemma 2.4, $(\beta_{-s}u)_1\in B(G/P_{\min};\lambda(-s))$. Now Lemma 2.2 and Theorem 2.3 imply that $(\beta_{-s}u)_1=0$. ∎

REMARK. It is also possible to avoid the use of Theorem 2.3: one can use methods from the second part of this chapter; the details are left to the reader.

Consider the representation π of G on $\mathfrak{g}_{\mathbb{C}}$, defined by

$$\pi(g)X=\mathrm{Ad}\,(g)X=gXg^{-1} \quad (g\in G,\ X\in\mathfrak{g}_{\mathbb{C}}).$$

LEMMA 7.2. A) *π is irreducible.*

B) *$\pi|_K$ splits up into two irreducible components for $n\neq 4$: $\mathfrak{g}_{\mathbb{C}}=\mathfrak{k}_{\mathbb{C}}\oplus\mathfrak{p}_{\mathbb{C}}$. For $n=4$, $\mathfrak{k}_{\mathbb{C}}$ again splits up into two minimal invariant subspaces.*

PROOF. A) This is an easy consequence of the fact that $sl(n,\mathbb{C})$ is simple.

B) Of course, $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ are K-invariant. If $n\neq 4$, $so(n,\mathbb{C})$ is simple, implying that $\mathfrak{k}_{\mathbb{C}}$ is minimal invariant. For $n=4$,

$$\begin{pmatrix} \begin{matrix}0 & -1\\ 1 & 0\end{matrix} & \Theta \\ \Theta & \begin{matrix}0 & -1\\ 1 & 0\end{matrix}\end{pmatrix} \quad\text{and}\quad \begin{pmatrix} \begin{matrix}0 & -1\\ 1 & 0\end{matrix} & \Theta \\ \Theta & \begin{matrix}0 & 1\\ -1 & 0\end{matrix}\end{pmatrix}$$

are easily seen to generate irreducible subspaces of $\mathfrak{k}_{\mathbb{C}}$. Now consider $\mathfrak{p}_{\mathbb{C}}$. Note that $H_0=E_{n1}+E_{1n}$ is the unique vector X in $\mathfrak{g}_{\mathbb{C}}$ — up to a scalar — with the following property:

$$\pi(k)X=\chi(k)X \text{ for all } k\in K\cap\bar{M}.$$

Suppose $U \subset \mathfrak{p}_{\mathbb{C}}$ is an invariant subspace. Then there is a K-invariant subspace V of $\mathfrak{p}_{\mathbb{C}}$ with $\mathfrak{p}_{\mathbb{C}} = U \oplus V$. Using H_0, one easily shows that either $H_0 \in U$ or $H_0 \in V$. Suppose $H_0 \in U$. It is well-known, that $\pi(K)H_0$ spans $\mathfrak{p}$, completing the proof. ■

For matrices $X, Y \in \mathfrak{g}_{\mathbb{C}}$, define $(X, Y) = \text{trace}\,(X\bar{Y})$. Let

$$X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} & & -1 \\ & 0 & \\ 1 & & \end{pmatrix} \in \mathfrak{k}$$

and define $Y \in A(K)$ by $Y(k) = (\pi(k)X_1, X_1)$ (cf. the proof of Lemma 6.4 and [Kosters, 9], p. 105 where $2Y$ is used).

LEMMA 7.3. *Let $u \in B(G/H; M_s)$. If $\beta_s u =$), then for all $g \in G$:*

$$\int_K (\beta_{-s} u)_2(gk) Y(k) dk = 0.$$

PROOF. Take an arbitrary u in $B(G/H; M_s)$ and $g \in G$. Define

$$V_g(g') = \int_K u(gk^{-1}g') Y(k) dk \quad (g' \in G).$$

As in the proof of Lemma 7.1, $V_g \in B(G/H; M_s)$. Let $\{X_1, \ldots, X_m\}$ be an orthonormal basis of $\mathfrak{k}$, with respect to the inner product $-(\cdot, \cdot)$. Here $m = \dim \mathfrak{k}$. Then for $k \in K$:

$$V_g(kg') = \int_K u(gk_0^{-1}g') Y(kk_0) dk_0 =$$

$$= \sum_{j=1}^{m} (X_j, \pi(k^{-1})X_1) \int_K u(gk_0^{-1}g')(\pi(k_0)X_1, X_j) dk_0$$

showing that V_g is K-finite. From [Varadarajan, 16], p. 310 it follows that V_g is real analytic. Furthermore, for $k \in K$ and $t \in \mathbb{R}$:

$$V_g(ka_t) = \sum_{j=1}^{m} (X_j, \pi(k^{-1})X_1) \int_K u(gk_0^{-1}a_t)(\pi(k_0)X_1, X_j) dk_0.$$

Note that $V_g(ka_t) = V_g(ka_t m) = V_g(kma_t)$ for $k \in K$, $t \in \mathbb{R}$ and $m \in K \cap M$, and that if $n \neq 4$, the space of vectors in $\mathfrak{k}$ fixed under the action of $K \cap M$, is one dimensional and spanned by X_1. Therefore, integration over $K \cap M$, which is projection onto this space, shows that only $j = 1$ remains in the summation. Thus for $n \neq 4$:

$$V_g(ka_t) = Y(k) V_g(a_t) \quad (k \in K,\ t \in \mathbb{R}).$$

Now remark that $Y \in B_2(K/K \cap M)$, and therefore it makes sense to compute $\mathscr{P}_s^2 Y$. The same arguments as above show that:

$$(\mathscr{P}_s^2 Y)(ka_t) = Y(k)(\mathscr{P}_s^2 Y)(a_t) \quad (k \in K,\ t \in \mathbb{R}).$$

Furthermore, both V_g and $\mathscr{P}_s^2 Y$ satisfy the differential equation (M_s). From this, it follows that $V_g(a_t)$ and $\mathscr{P}_s^2 Y(a_t)$ satisfy

$$f''(t)+(2(n-2)cth2t+2th2t)f'(t)-\left(\frac{4(n-2)}{\mathrm{sh}^2\, 2t}+s^2-\varrho^2\right)f(t)=0.$$

It is well-known that this differential equation has a unique analytic solution — up to a scalar —, which is an odd function in t (cf. the proof of Lemma 7.5). Note that $(\mathscr{P}_s^2 Y)(a_t)$ and $V_g(a_t)$ are odd in t, which easily follows from $w_0 a_t w_0 = a_{-t} (t \in \mathbb{R})$, where

$$w_0=\begin{pmatrix} -1 & & & & \\ & -1 & & \Theta & \\ & & 1 & & \\ & \Theta & & \ddots & \\ & & & & 1 \end{pmatrix} \in K \cap H.$$

Remark that $(\mathscr{P}_s^2 Y)(a_t)$ is a nonzero function: otherwise $\mathscr{P}_s^2 Y$ would be zero, which contradicts Lemma 5.5.

Summarizing these facts: for some $\upsilon(g)\in\mathbb{C}$:

$$V_g=\upsilon(g)\mathscr{P}_s^2 Y.$$

Now suppose that $\beta_s u=0$. Arguing as in the proof of Lemma 7.3, it follows that for all $g\in G$:

$$\int_K (\beta_{-s}u)(gk)Y(k)dk=0.$$

Using the fact that

$$\int_K (\beta_{-s}u)_1(gk)Y(k)dk=0 \quad (g\in G)$$

for all $u\in B(G/H;M_s)$, the lemma follows for $n\neq 4$. Now let $n=4$. Then the matrix

$$X_0=\frac{1}{\sqrt{2}}\begin{pmatrix} \Theta & & & 0 \\ & & -1 & \\ & 1 & & \\ 0 & & & \Theta \end{pmatrix}$$

is also fixed by $K\cap M$. Let $Z(k)=(\pi(k)X_0,X_1)$ $(k\in K)$. The same argument as above shows that

$$V_g(ka_t)=Z(k)\Phi_1(t)+Y(k)\Phi_2(t) \quad (k\in K,\ t\in\mathbb{R})$$

where Φ_1 and Φ_2 are real analytic solutions of the differential equation mentioned before. Let $\Psi(t)$ be the nonzero real analytic solution of this equation. Then, for some $\lambda_1,\lambda_2,\mu_1,\mu_2\in\mathbb{C}$:

$$(\mathscr{P}_s^2 Y)(ka_t)=(\lambda_1 Y(k)+\lambda_2 Z(k))\Psi(t),$$

$$(\mathscr{P}_s^2 Z)(ka_t)=(\mu_1 Y(k)+\mu_2 Z(k))\Psi(t),$$

for $k \in K$ and $t \in \mathbb{R}$. According to Lemma 5.5, $\mathscr{P}_s^2 Y$ and $\mathscr{P}_s^2 Z$ are linearly independent in $B(G/H;M_s)$. Therefore, V_g is a linear combination of $\mathscr{P}_s^2 Y$ and $\mathscr{P}_s^2 Z$, showing that

$$\beta_s V_g = \upsilon_1(g) c_2(s) Y + \upsilon_2(g) c_2(s) Z$$

for certain complex numbers $\upsilon_1(g)$ and $\upsilon_2(g)$. It follows that

$$\beta_s V_g(e) = -\upsilon_1(g) c_2(s) = \int_K (\beta_s u)(gk) Y(k) dk,$$

$$\beta_s V_g \left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right) = \upsilon_2(g) c_2(s).$$

So, if $\beta_s u = 0$, then $\upsilon_1(g) = \upsilon_2(g) = 0$, implying $V_g = 0$. The proof is easily completed now. ∎

In the previous lemma we considered the restriction of π to K and in particular the action on $\mathfrak{k}_{\mathbb{C}}$. Now we look at $\mathfrak{p}_{\mathbb{C}}$. Let

$$X_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix}, \quad X_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \text{ and}$$

$$X_4 = \sqrt{2/n(n-2)} \begin{pmatrix} \frac{n-2}{2} & & & & 0 \\ & -1 & & & \\ & & \ddots & & \\ & & & -1 & \\ 0 & & & & \frac{n-2}{2} \end{pmatrix}.$$

Then $\{X_2, X_3, X_4\}$ is an orthonormal basis of the set of vectors in $\mathfrak{p}$ that are fixed under $K \cap M$. Note that $\pi(w)X_2 = X_2$, $\pi(w)X_3 = -X_3$ and $\pi(w)X_4 = X_4$; with w_0 as in the proof of Lemma 7.3, $\pi(w_0)X_2 = -X_2$, $\pi(w_0)X_3 = X_3$ and $\pi(w_0)X_4 = X_4$. Define $F_{ij}(k) = (\pi(k)X_i, X_j)$, for $i, j \in \{2,3,4\}$.

Then $F_{ij} \in A(K)$. The crucial lemma in this chapter is:

LEMMA 7.4. *Let $u \in B(G/H;M_s)$. If $\beta_s u = 0$, then for all $g \in G$:*

$$\int_K (\beta_{-s} u)_2(gk) F_{33}(k) dk = 0.$$

Copying the proof of Lemma 7.3 we note that

$$(\mathscr{P}_s^{\varepsilon(i)} F_{ij})(ka_t) = \sum_{l=2,3,4} F_{lj}(k) \Phi_{il}(t), \quad i, j \in \{2,3,4\}$$

where $\varepsilon(2)=\varepsilon(4)=1$ and $\varepsilon(3)=2$ and

$$\Phi_{il}(t)=\int_K P_s^{\varepsilon(i)}(k^{-1}a_t)F_{il}(k)dk.$$

LEMMA 7.5. *The functions $\Phi_{il}(t)$ $(i,l=2,3,4)$ satisfy the following differential equations:*

$$(1)\qquad D\Phi_{i2}+\left\{\frac{-(n-2)}{\mathrm{sh}^2\, t}+\frac{(n-2)}{\mathrm{ch}^2\, t}+\frac{4}{\mathrm{ch}^2\, 2t}-(s^2-\varrho^2)\right\}\Phi_{i2}=0$$

$$(2)\qquad D\Phi_{i3}-\left\{\frac{4(n-2)}{\mathrm{sh}^2\, 2t}-\frac{4}{\mathrm{ch}^2\, 2t}+s^2-\varrho^2\right\}\Phi_{i3}+4\sqrt{n(n-2)}\,\frac{\mathrm{ch}\, 2t}{\mathrm{sh}^2\, 2t}\,\Phi_{i4}=0$$

$$(3)\qquad D\Phi_{i4}-\left\{\frac{4n}{\mathrm{sh}^2\, 2t}+s^2-\varrho^2\right\}\Phi_{i4}+4\sqrt{n(n-2)}\,\frac{\mathrm{ch}\, 2t}{\mathrm{sh}^2\, 2t}\,\Phi_{i3}=0.$$

Here $D\Phi_{il}=\Phi_{il}''+(2(n-2)cth2t+2th2t)\Phi_{il}'$.

PROOF. Note that $\square\mathscr{P}_s^{\varepsilon(i)}F_{ij}=(s^2-\varrho^2)F_{ij}$ $(i,j=2,3,4)$ and use [Kosters, 9], Lemma 2.4.2 and p. 109 to show that

$$(s^2-\varrho^2)\sum_{l=2,3,4}F_{lj}(k)\Phi_{il}(t)=$$

$$=\left\{\frac{-(n-2)}{\mathrm{sh}^2\, t}+\frac{(n-2)}{\mathrm{ch}^2\, t}+\frac{4}{\mathrm{ch}^2\, 2t}\right\}F_{2j}(k)\Phi_{i2}(t)+F_{2j}(k)D\Phi_{i2}(t)+$$

$$+\left\{\frac{\sqrt{2}}{\mathrm{sh}^2\, t}\left[\frac{(2-n)}{\sqrt{2}}\,F_{3j}(k)+\sqrt{n(n-2)/2}\;F_{4j}(k)\right]-\right.$$

$$-\frac{\sqrt{2}}{\mathrm{ch}^2\, t}\left[\frac{(2-n)}{\sqrt{2}}\,F_{3j}(k)-\sqrt{n(n-2)/2}\;F_{4j}(k)\right]+$$

$$\left.+\frac{4}{\mathrm{ch}^2\, 2t}\,F_{3j}(k)\right\}\Phi_{i3}(t)+F_{3j}(k)D\Phi_{i3}(t)+$$

$$+\left\{\frac{1}{\mathrm{sh}^2\, t}\sqrt{2n/(n-2)}\left[\frac{(n-2)}{\sqrt{2}}\,F_{3j}(k)-\sqrt{n(n-2)/2}\;F_{4j}(k)\right]-\right.$$

$$\left.-\frac{1}{\mathrm{ch}^2\, t}\sqrt{2n/(n-2)}\left[\frac{(2-n)}{\sqrt{2}}\,F_{3j}(k)-\sqrt{n(n-2)/2}\;F_{4j}(k)\right]\right\}\Phi_{i4}(t)+$$

$$+F_{4j}(k)D\Phi_{i4}(t).$$

Using the fact that $\{F_{2j},F_{3j},F_{4j}\}$ is a linearly independent set in $A(K)$, this equation implies the lemma. ∎

LEMMA 7.6. *Notation as in Lemma 7.5.*

i) *Up to a scalar, equation (1) has a unique odd real analytic solution on $\mathbb{R}$. The only even analytic solution is 0.*

ii) *The space of analytic functions (f,g) on $\mathbb{R}$, that are even and satisfy (2) and (3), is two dimensional. The only odd analytic solution is $(0,0)$.*

PROOF. We only prove ii), the proof of i) being easier. Suppose that $f(t) = a_0 + a_2t^2 + \ldots$ and $g(t) = b_0 + b_2t^2 + \ldots$ give an analytic solution of (2) and (3), for t sufficiently near 0. It is easily seen that the lowest degree term in (2) or (3) gives the equation

$$-(n-2)a_0 + \sqrt{n(n-2)}\,b_0 = 0.$$

Assuming that $a_0 = 0$, we get $b_0 = 0$. Again we consider the lowest degree term in (2) and (3), which is the constant term now. This implies

$$-\sqrt{n}\,a_2 = \sqrt{n-2}\,b_2.$$

Therefore, if $a_2 = 0$ then $b_2 = 0$.

Proceeding by induction, we suppose that $a_0 = a_2 = \ldots = a_{2l-2} = 0$, $b_0 = b_2 = \ldots = b_{2l-2} = 0$ for some l in $\{2, 3, \ldots\}$. The lowest degree term gives:

$$\begin{cases} (2l(2l-1) + (n-2)2l - (n-2))a_{2l} + \sqrt{n(n-2)}\,b_{2l} = 0 \\ (2l(2l-1) + (n-2)2l - n)b_{2l} + \sqrt{n(n-2)}\,a_{2l} = 0. \end{cases}$$

The determinant of this system is equal to $4l(l-1)(2l+n-3)(2l+n-1)$, which is only zero for $l = 0, 1, \frac{1}{2}(-n+3)$ and $\frac{1}{2}(-n+1)$. Using $l \geqq 2$, we have $a_{2l} = b_{2l} = 0$.

Summarizing these facts a_0 and a_2 determine the pair (f, g), so the solution space is at most two dimensional. The fact that there are two linearly independent solutions will be proved during the proof of Lemma 7.4. In order to prove the second statement of ii), suppose that $f(t) = a_1t + a_3t^3 + \ldots$ and $g(t) = b_1t + b_3t^3 + \ldots$ give a solution of (2) and (3), for t sufficiently near 0. The lowest degree term in (2) and (3) has to be zero, and therefore

$$\begin{cases} \sqrt{n(n-2)}\,b_1 = 0 \\ -2b_1 + \sqrt{n(n-2)}\,a_1 = 0 \end{cases}$$

whence $a_1 = b_1 = 0$. Now we assume that $a_1 = a_3 = \ldots = a_{2l-1} = 0$ and $b_1 = b_3 = \ldots = b_{2l-1} = 0$ for some l in $\{1, 2, 3, \ldots\}$. Compute the coefficients of t^{2l-1} in (2) and (3). This yields

$$\begin{cases} ((2l+1)2l + (n-2)(2l+1) - (n-2))a_{2l+1} + \sqrt{n(n-2)}\,b_{2l+1} = 0 \\ ((2l+1)2l + (n-2)(2l+1) - n)b_{2l+1} + \sqrt{n(n-2)}\,a_{2l+1} = 0. \end{cases}$$

The determinant of this system is equal to $(2l+1)(2l-1)(2l+n-2)(2l+n)$, which is nonzero because $l \geqq 1$. So $a_{2l+1} = b_{2l+1} = 0$, thereby showing that $f = g = 0$. ■

REMARK. It is also possible to prove this lemma by using some standard techniques for the solution of these types of differential equations. These techniques can be found in most books on linear differential equations.

Now we can give the

PROOF OF LEMMA 7.4. Take an arbitrary $u \in B(G/H; M_s)$ and $g \in G$. Consider for $g' \in G$:

$$V_g(g') = \int_K u(gk^{-1}g')F_{33}(k)dk.$$

As before, V_g is a real analytic element of $B(G/H; M_s)$ (cf. the proofs of Lemma 7.1 and Lemma 7.3). We have

$$V_g(ka_t) = \sum_{l=2,3,4} F_{l3}(k)U_l(t) \quad (k \in K,\ t \in \mathbb{R})$$

where

$$U_l(t) = \int_K u(gk^{-1}a_t)F_{3l}(k)dk \quad (t \in \mathbb{R}).$$

With notation as in Lemma 7.5 we see that U_2 satisfies equation (1), and the pair (U_3, U_4) satisfies (2) and (3). Note that U_2 is odd whereas U_3 and U_4 are even. This follows from

$$V_g(ka_tw_0) = V_g(kw_0a_{-t}) = V_g(ka_t) \quad (k \in K,\ t \in \mathbb{R})$$

with w_0 as in the proof of Lemma 7.3. We remark that $\{\mathscr{P}_s^1F_{23}, \mathscr{P}_s^2F_{33}, \mathscr{P}_s^1F_{43}\}$ is a linearly independent set in $A(X)$, which is an easy consequence of Lemma 5.5. Now suppose that Φ spans the space of odd analytic solutions of (1) and that $\{(\Psi_1, \Psi_2), (\Sigma_1, \Sigma_2)\}$ spans the space of even analytic pairs that satisfy (2) and (3). Consider the space V spanned by

$$\{F_{23}(k)\Phi(t), F_{33}(k)\Psi_1(t) + F_{43}(k)\Psi_2(t), F_{33}(k)\Sigma_1(t) + F_{43}(k)\Sigma_2(t)\}$$

Of course, the dimension of V is at most three. The fact that, for $i = 2, 3, 4$, $\mathscr{P}_s^{\varepsilon(i)}F_{i3} \in V$ shows that the dimension of V indeed is equal to three (cf. the proof of Lemma 7.3). It is easily seen that V_g is an element of V, and therefore

$$V_g = \upsilon_2(g)\mathscr{P}_s^1F_{23} + \upsilon_3(g)\mathscr{P}_s^2F_{33} + \upsilon_4(g)\mathscr{P}_s^1F_{43}$$

for certain complex numbers $\upsilon_i(g)$ $(i = 2, 3, 4)$. So:

$$\beta_sV_g(e) = \upsilon_3(g)c_2(s) = \int_K (\beta_su)(gk)F_{33}(k)dk.$$

If $\beta_su = 0$, then $\upsilon_3(g) = 0$. Using the matrix C, introduced in Chapter 2, we derive

$$\beta_sV_g(C) = \upsilon_2(g)c_1(s) = \int_K (\beta_su)(gk)F_{33}(Ck)dk$$

from which $\upsilon_2(g) = 0$ easily follows. Substitution of

$$\begin{pmatrix} 0 & -1 & & \Theta \\ 1 & 0 & & \\ & & 1 & & \\ & & & \ddots & \\ \Theta & & & & 1 \end{pmatrix}$$

yields that $v_4(g)=0$. Therefore, $V_g=0$, whence

$$\beta_{-s}V_g(e)=0=\int_K (\beta_{-s}u)(gk)F_{33}(k)dk.$$

This integral is equal to

$$\int_K (\beta_{-s}u)_2(gk)F_{33}(k)dk$$

thereby completing the proof of the lemma. ■

Note that this also completes the proof of Lemma 7.6. Define for $g\in G$:

$$F(g)=(\pi(g)(X_3+X_1),X_3-X_1).$$

Then $F(k)=F_{33}(k)-Y(k)$ $(k\in K)$, because $(\pi(k)X_1,X_3)=0$ for all $k\in K$. Note that $X_3+X_1\in\mathfrak{g}(2\alpha_0)$ and $X_3-X_1\in\mathfrak{g}(-2\alpha_0)$. As a consequence of Lemma 7.3 and Lemma 7.4 we have

COROLLARY 7.7 *Let $u\in B(G/H;M_s)$. If $\beta_s u=0$, then for all $g\in G$:*

$$\int_K (\beta_{-s}u)_2(gk)F(k)dk=0.$$ ■

Using Lemma 2.5, it is easy to see that $(\beta_{-s}u)_2\in B(G/P_{\min};L_{\lambda(-s)};\chi)$. Note that F is real analytic, $F(e)=2$, and

$$F(\bar{n}gman)=\chi(m)e^{\alpha_{1n}(\log a)}F(g)$$

for all $g\in G$, $\bar{n}\in\bar{N}_{\min}=\theta(N_{\min})$, $m\in M_{\min}$, $a\in A_{\min}$ and $n\in N_{\min}$. Therefore, using a well-known integration formula (cf. [Oshima, Sekiguchi, 13], p. 51), we have

$$\int_K (\beta_{-s}u)_2(gk)F(k)dk=\int_K (\beta_{-s}u)_2(k)F(g^{-1}k)e^{-(\lambda(-s+2)+\varrho_{\mathfrak{p}})(H(g^{-1}k))}dk$$

where $H(g^{-1}k)\in\mathfrak{a}_{\mathfrak{p}}$ is such that $g^{-1}k\in K\exp(H(g^{-1}k))N_{\min}$. As in [Wallach, 17], define:

$$1_\lambda(g)=e^{-\lambda(H(g))}\quad (g\in G,\ \lambda\in\mathfrak{a}^*_{\mathfrak{p},\mathbb{C}})$$

and write $(l_g f)(x)=f(gx)$ for $g,x\in G$ and $f\in A(G)$. With these notations we have:

$$\langle(\beta_{-s}u)_2,l_g(F1_{\lambda(-s+2)+\varrho_{\mathfrak{p}}})\rangle=0 \qquad (*)$$

for all $g\in G$. We want to show that $(\beta_{-s}u)_2=0$. Define

$$B_\chi(K/M_{\min})=\{f\in B(K)\,|\,f(km)=\chi(m)f(k)\text{ for all }K\in K,\ m\in M_{\min}\}$$

$$L^2_\chi(K/M_{\min})=\{f\in L^2(K)\,|\,f(km)=\chi(m)f(k)\text{ for (almost) all}$$
$$k\in K,\ m\in M_{\min}\}$$

$$A_\chi(K/M_{\min})=A(K)\cap L^2_\chi(K/M_{\min}).$$

Then $B_\chi(K/M_{\min})$ is the topological dual of $A_\chi(K/M_{\min})$ where the spaces are topologized in the familiar way. Note that $L^2_\chi(K/M_{\min})$ is a representation space for the principal series representation $\pi^{\chi,\lambda}$ ($\lambda \in \mathfrak{a}^*_{\mathfrak{p},\mathbb{C}}$), by

$$(\pi^{\chi,\lambda}(g)f)(k) = f(\kappa_0(g^{-1}k))e^{-\lambda(\log(H(g^{-1}k)))}$$

for $g \in G$, $k \in K$ and $f \in L^2_\chi(K/M_{\min})$; here we defined $\kappa_0(g^{-1}k) \in K$ and $H(g^{-1}k) \in \mathfrak{a}_\mathfrak{p}$ by:

$$g^{-1}k \in \kappa_0(g^{-1}k) \exp(H(g^{-1}k))N_{\min}.$$

In the same way, $L^2(K/M_{\min})$ becomes a representation space for the spherical principal series representation π^λ (cf. [Wallach, 17], Chapter 8). We need

LEMMA 7.8. *Let $\nu \in \mathfrak{a}^*_{\mathfrak{p},\mathbb{C}}$ and $\mu \in \mathfrak{a}^*$ be such that:*

1) *1_ν is a cyclic vector for the representation π^ν on $L^2(K/M_{\min})$.*
2) *$L^2_\chi(K/M_{\min})$ contains a nonzero finite dimensional $\pi^{\chi,\mu}$-invariant subspace V_μ.*

Suppose that W is an $\bar{N}_{\min}$-invariant function in V_μ, $W \neq 0$. Then $1_\nu W$ is a cyclic vector for the representation $\pi^{\chi,\nu+\mu}$.

PROOF. The lemma is a special case of Lemma 8.13.9 from [Wallach, 17]. ■

In order to show that (*) implies $(\beta_{-s}u)_2 = 0$, it suffices to prove

LEMMA 7.9. *Let $s \in \mathbb{C}$, $2s \notin \mathbb{Z}$. Then $F1_{\lambda(-s+2)+\varrho_\mathfrak{p}}$ is a cyclic vector for the left regular representation on $A_\chi(K/M_{\min})$, i.e. its left translates span a dense subspace of $A_\chi(K/M_{\min})$.*

PROOF. We use the previous lemma, with $\mu = -\alpha_{1n}$ and $\nu = \lambda(-s+2) + \varrho_\mathfrak{p}$. We have to show that 1_ν is a cyclic vector for the representation π^ν. This follows from [Helgason, 5], p. 114 and [Helgason, 6], p. 198, where it is proved that 1_ν is cyclic if and only if $e(\nu - \varrho_\mathfrak{p}) \neq 0$. Here $e(\nu - \varrho_\mathfrak{p}) = e(\lambda(-s+2))$, which is nonzero because of Lemma 2.2. and the fact that $2s \notin \mathbb{Z}$. Now Lemma 7.8 implies that $F1_{\lambda(-s+2)+\varrho_\mathfrak{p}}$ is a cyclic vector for the representation $\pi^{\chi,\lambda(-s)+\varrho_\mathfrak{p}}$. Note that F satisfies the properties mentioned in Lemma 7.8. Therefore, the left translates of $F1_\nu$ span a dense subspace, say V, of $L^2_\chi(K/M_{\min})$. Write $L = L^2_\chi(K/M_{\min})$ and $B = A_\chi(K/M_{\min})$. We have to prove that V is dense in B. Note that the K-finite vectors in L and B constitute dense subspaces in these spaces.

First we remark that the K-finite vectors in L are automatically real analytic functions, because they are linear combinations of matrix coefficients of irreducible unitary representations of K. Finally we show that V contains the K-finite vectors of L. Therefore, let P_δ be the projection of L onto the space L_δ of K-finite vectors of type δ, for fixed $\delta \in \hat{K}$. P_δ is continuous. L_δ is finite dimensional, for example because of Frobenius reciprocity. Then

$$L_\delta = P_\delta(L) = P_\delta \text{ (closure of } V) \subset \text{closure of } P_\delta(V) = P_\delta(V) \subset L_\delta$$

whence $L_\delta = P_\delta(V) = V_\delta$, the set of vectors of type δ in V.

Combining these facts, we see that V contains a dense subspace of B. ■

Therefore, if $u \in B(G/H;M_s)$ satisfies $\beta_s u = 0$, then $(\beta_{-s}u)_2 = 0$. Together with Lemma 7.1. this implies that $\beta_{-s}u = 0$, thereby proving Lemma 5.6. This completes the proof of Theorem 5.4.

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