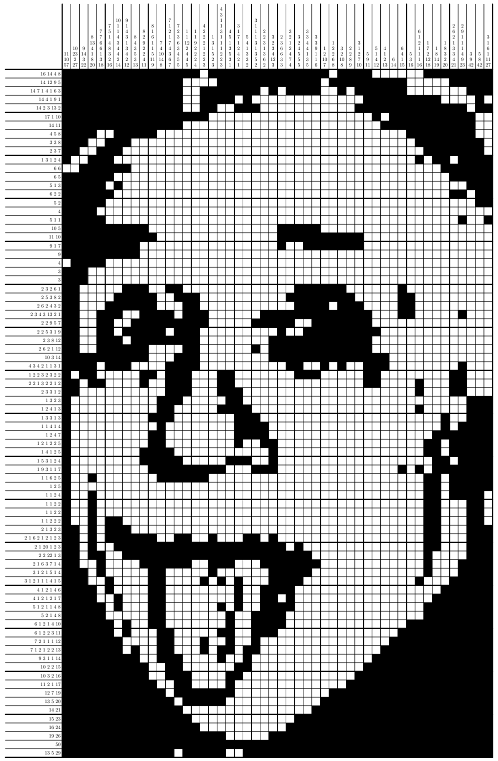
Some Aspects of Discrete Tomography

Walter Kosters, Universiteit Leiden



Wednesday April 13, 2011; Vlijmen

`www.liacs.nl/home/kosters/`

When talking about Japanese puzzles, everyone thinks of Sudoku.

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

When talking about Japanese puzzles, everyone thinks of Sudoku.

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

source: Wikipedia

But we will talk about Nonograms today.

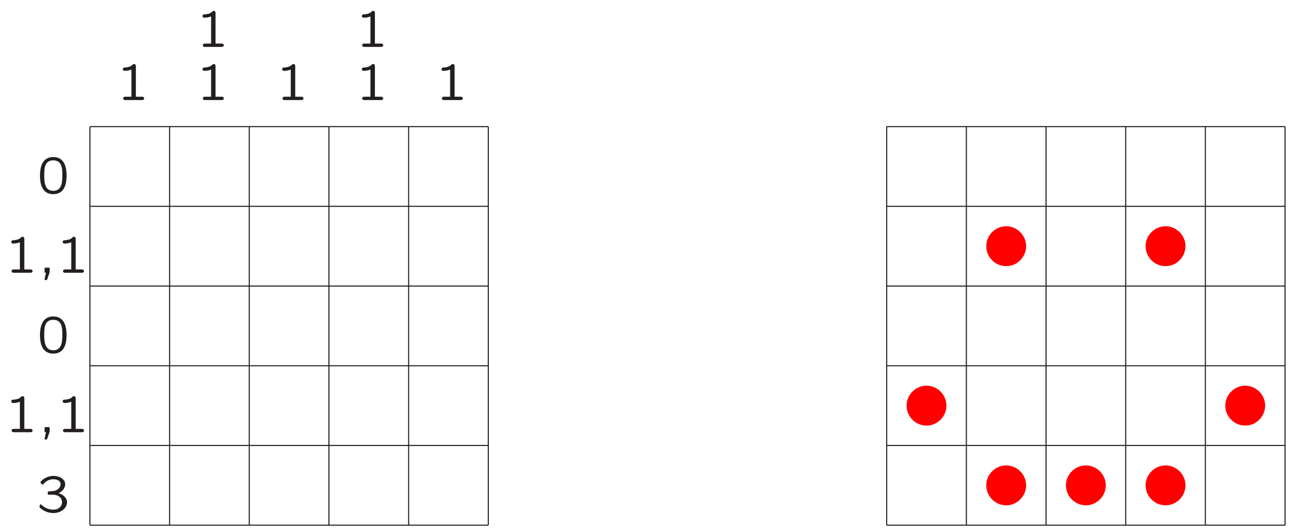
A Nonogram is a puzzle; a small example:

		1		1	
	1	1	1	1	1
0					
1,1					
0					
1,1					
3					

Next to each row and column we enumerate the lengths of consecutive series of red pixels.

Where are these red = **black** pixels?

The (unique) solution looks like this:



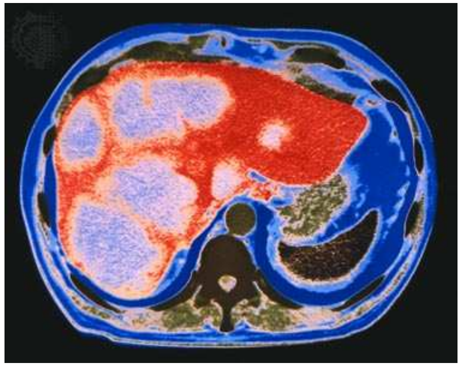
Next to each row and column we enumerate the lengths of consecutive series of red pixels — in order.

Why are scientists interested in Nonograms?

Tomography tries to solve the following problem:

How to reconstruct an object from projections?

Examples:



- Solve Nonograms

- How do we look like, given CT-scans? (Computerized Tomography = CT $\supseteq$ DT = Discrete Tomography)

- Where are the "holes" in a diamond?

In Discrete Tomography we try to reconstruct an object from its projections.

An object "is" a finite subset of $\mathbf{Z}^2$ (so integer points in 2D space).

A projection gives *all* relevant "line sums" over lines parallel to a given line, e.g., all horizontal lines and all vertical lines (2 projections). It is also possible to use all lines through a given point.

In CT one typically has *many* projections, in DT a few.

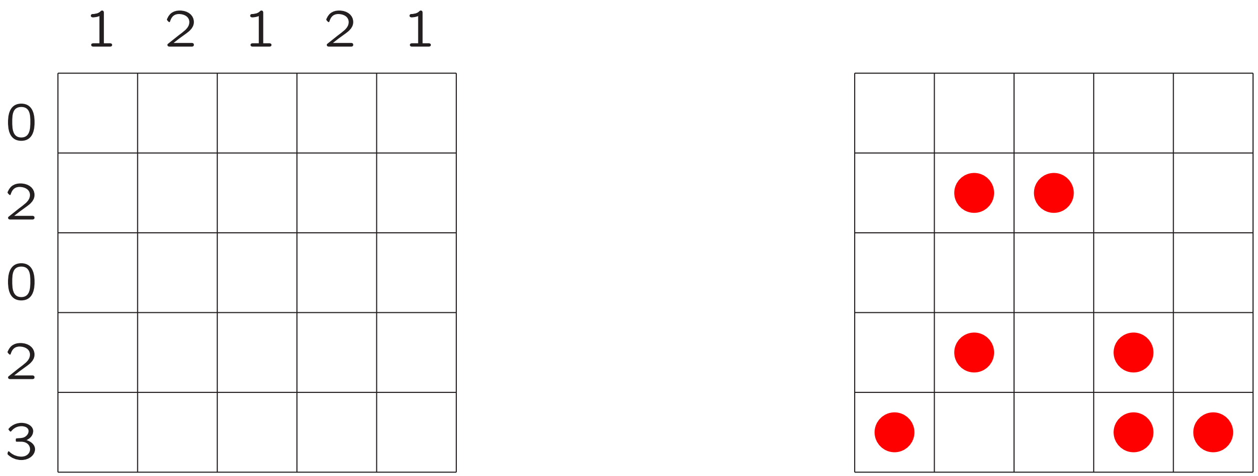
A small example of a Discrete Tomography problem:

	1	2	1	2	1
0					
2					
0					
2					
3					

Next to each row and column we give the *total* number of red pixels.

Where are these red = **black** pixels?

A (non-unique) solution looks like this:



Next to each row and column we give the (total) number of red pixels.

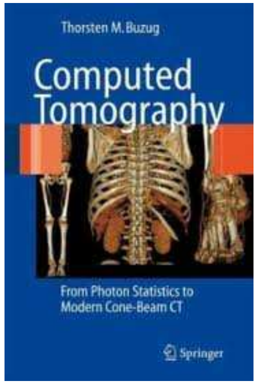
The problem can be defined more general:

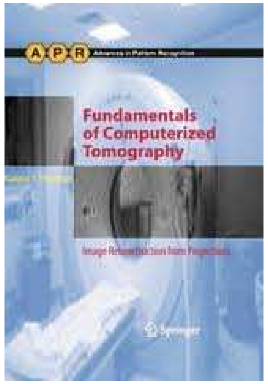
> Given an unknown function f on some domain D (discrete, or just some subset of $\mathbf{R}^n$), with a *discrete* range $\subseteq \mathbf{R}$, the task is to (approximately) reconstruct f, given sums (integrals) over certain subsets of D.

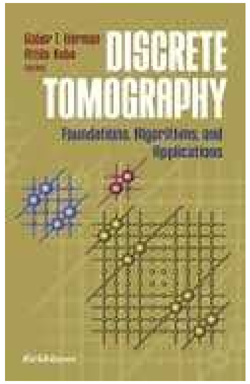
In our case, the range is $\{0,1\} = \{\text{white}, \textbf{black}\}$.

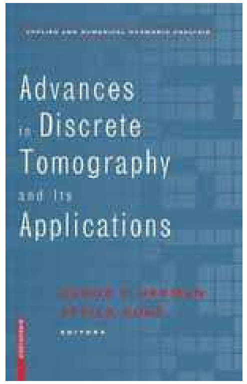
T.M. Buzug, Computed Tomography, Springer, 2008.

G.T. Herman, Fundamentals of Computerized Tomography, Springer 2009.









G.T. Herman & A. Kuba, Discrete Tomography, Foundations, Algorithms and Applications, Birkhäuser, 1999.

G.T. Herman & A. Kuba, Advances in Discrete Tomography and its Applications, Birkhäuser, 2007.

Usually we have three tasks:

Consistency Does an object with the given projection values (a "solution to the puzzle") exist?

Uniqueness Suppose there is a solution. Does there exist another one?

Reconstruction Construct a solution.

These problems, with horizontal and vertical projections, are solved in polynomial time by Ryser's Theorem from 1957.

But the problems for 3 or more projections are NP-hard!

$$\boxed{2 \neq 3}$$

If you were to use a flashlight in 2D, it would flicker like when throwing a stone into the water.

Ryser's Theorem

Given vectors $R = (r_1, \ldots, r_m)$ (all $\leq n$) and $S = (s_1, \ldots, s_n)$ (all $\leq m$) with the same total sum: $\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j$.

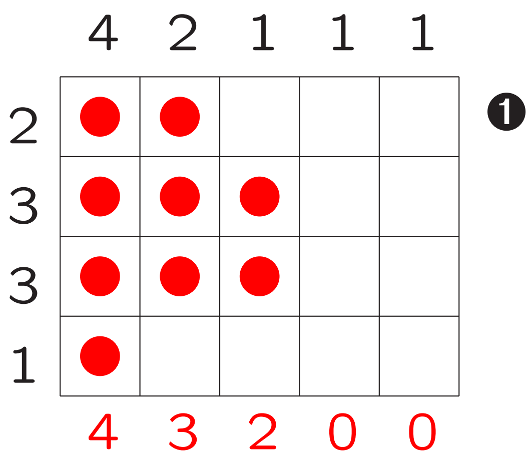
There is a 0–1 $m \times n$ matrix with row sums R and column sums S

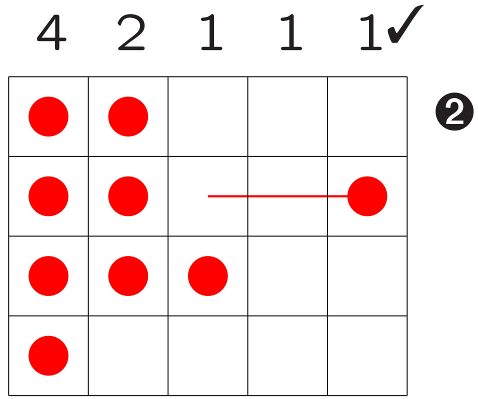
$$\Longleftrightarrow$$

for all ℓ with $2 \leq \ell \leq n$ we have: $\sum_{j=\ell}^{n} s'_j \geq \sum_{j=\ell}^{n} \overline{s}_j$.

Here the s'_j are the (non-increasing) sorted s_j, and the $\overline{s}_j$ are the column sums of *the* matrix with the r_i as row sums, and ones in the leftmost positions.

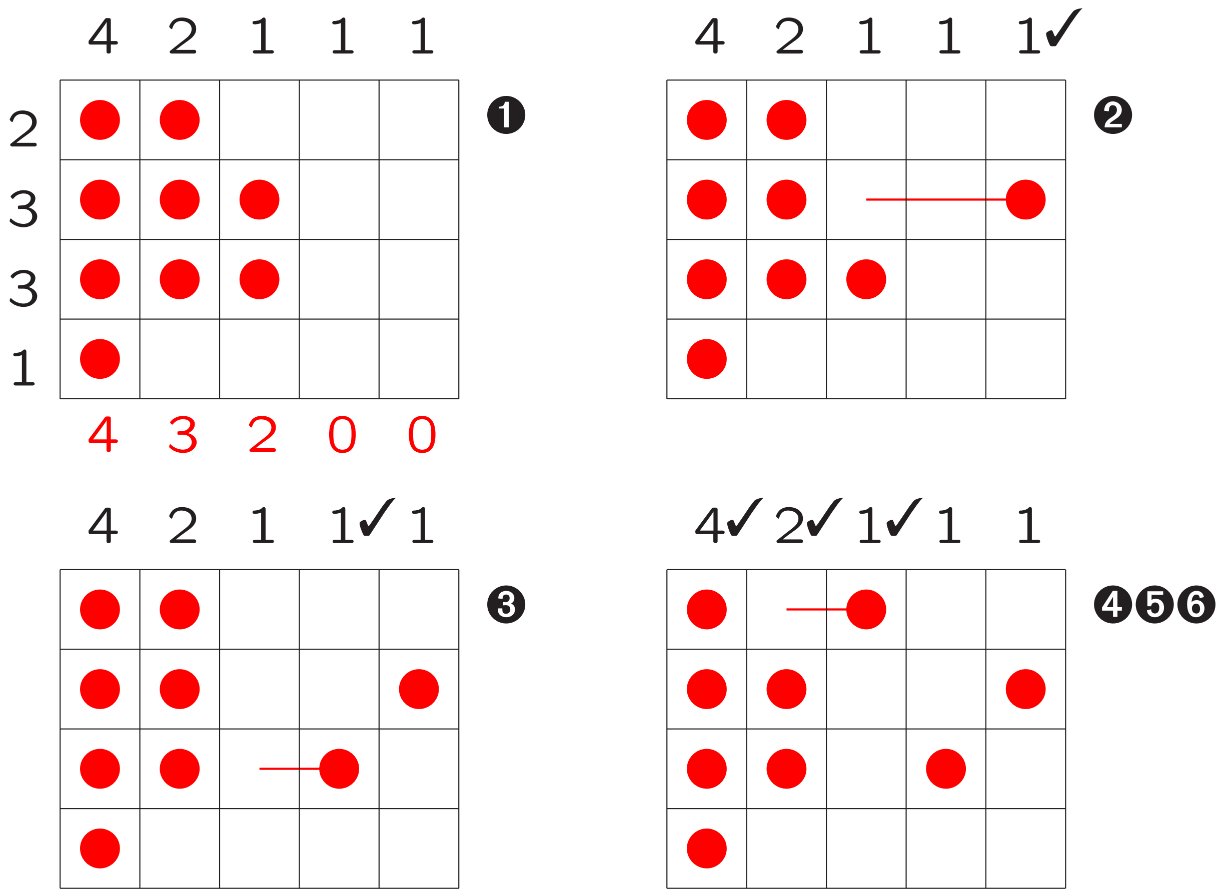
Ryser's algorithm constructs a solution in the following way. The column sums are already sorted in non-increasing order ($s = s'$):





First initialize each row as far to the left as possible: $\overline{s}$.
Then, from the right column backward, pull in red pixels from the left as needed.

Ryser's algorithm constructs a solution working backward from the last column:

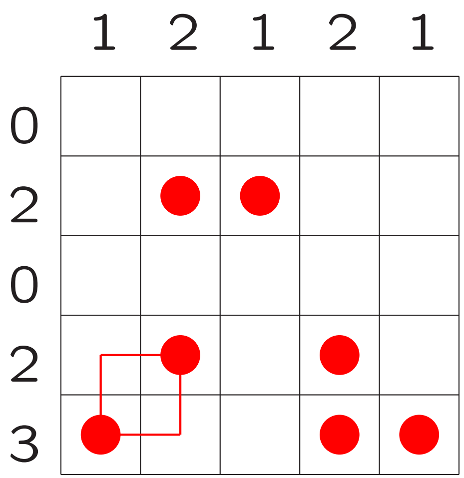


Ryser's Theorem (continued)
Furthermore, all solutions can be obtained from one another by a series of "switchings" using so-called switching components:

$$\boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \longleftrightarrow \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}}$$

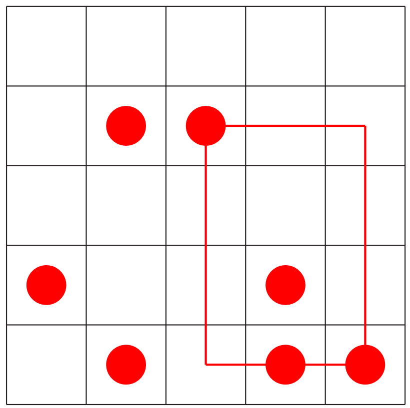
Note that a switch does not change row and column sums.

Open problem: what is the diameter of the solution graph?

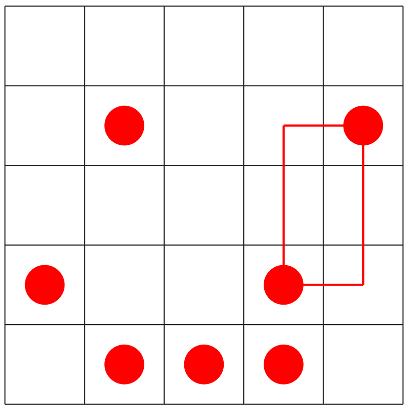


❶

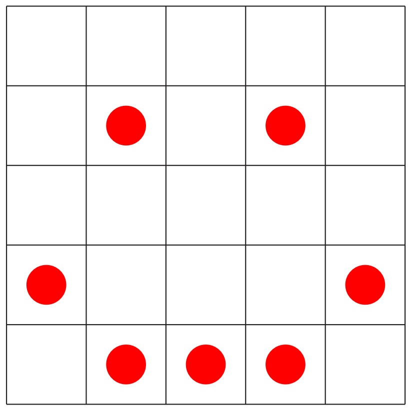
(slide 9)



❷



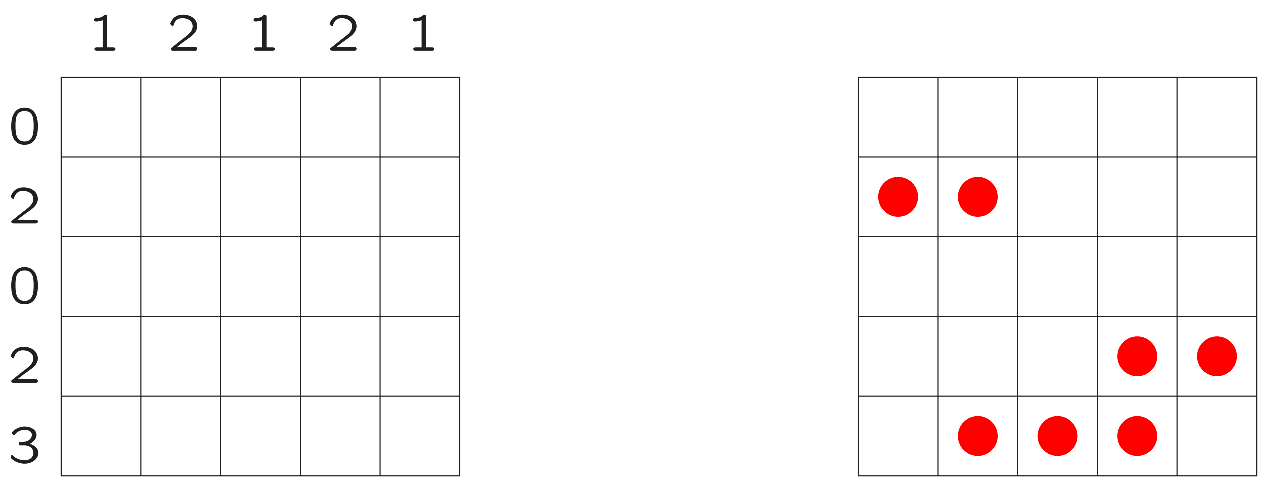
❸



❹

(slide 5)

In an h-convex object all rows must consist of *consecutive* red pixels: the rows have the “Nonogram property”.



For h-convex objects the 2 projections Consistency problem is NP-complete . . .

Now back to Nonograms: how to solve them?

Most humans use logic rules, combined with heuristics like "interchange row reasoning and column reasoning".

An example of such a logic rule is: "if the number 3 is next to a row/column of width 5, the middle pixel must be red". In this particular rule one looks at one row or column at a time.

Suppose you already know:

3,2,1	?	●	?	●	?	•	?	?	?	?	?

A • means a known white/empty pixel, a ● denotes a known filled pixel. The rest is still unknown.

Remember that we enumerate the lengths of consecutive series of red = **black** pixels — in order.

What can we conclude?

We conclude that for this row:

3,2,1	•	●	●	●	•	•	?	●	?	?	?

A • means a known white/empty pixel, a ● denotes a known filled pixel. The rest is still unknown.

So by examining a single row or column we can make progression.

How can a computer program draw such conclusions?

A first option is to use brute-force: try “all” possibilities. But a 5×5 Nonogram has

$$2^{25} = 2^{10} \cdot 2^{10} \cdot 2^{5} = 1024 \cdot 1024 \cdot 32 \approx 32 \text{ million}$$

possible solutions! And the “80×50 Einstein” from slide 1 has $2^{4000} \approx 10^{1200}$ possibilities.

So . . . no way! (But for small parts it might work.) We therefore first try some logic reasoning for a single line = row or column.

So we want a string $s_1 s_2 \ldots s_\ell$ over $\{\texttt{?}, \bullet, \text{●}\}$ to match a regular expression like $0^*1^30^+1^20^+1^10^*$, representing the Nonogram description 3,2,1.

We define $Fix(i,j)$ to be **`true`** if and only if the prefix $s_1 s_2 \ldots s_i$ can be made to match the first j elements from the description by "fixing" `?`'s to elements from $\{\bullet, \text{●}\}$.

Now we compute, using Dynamic Programming:

with $Fix(0, j-1), Fix(1, j-1), \ldots, Fix(i-1, j-1)$
somehow compute $Fix(i,j)$
and keep track of the "fixes".

So for a single line one can use Dynamic Programming.

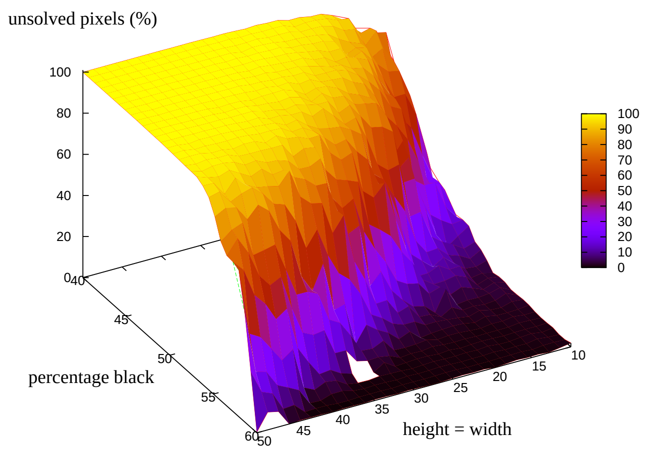
We want a string $s_1 s_2 \ldots s_\ell$ over the alphabet $\Sigma \cup \{?\}$ to match a regular expression $d_1 d_2 \ldots = \sigma_1\{a_1, b_1\}\sigma_2\{a_2, b_2\}\ldots$ (so first between a_1 and b_1 times the character σ_1, ...) in the following sense: $\mathit{Fix}(i,j)$ is `true` if and only if the prefix $s_1 s_2 \ldots s_i$ can be made to match $d_1 d_2 \ldots d_j$ by "fixing" ?'s to elements from Σ (e.g., $\Sigma = \{\bullet, \text{●}\}$):

$$\mathit{Fix}(i,j) = \bigvee_{p = \max(i - b_j, A_{j-1}, L_i^{\sigma_j}(s))}^{\min(i - a_j, B_{j-1})} \mathit{Fix}(p, j-1)$$

Here $A_j = \sum_{p=1}^{j} a_p$, $B_j = \sum_{p=1}^{j} b_p$ and $L_i^{\sigma}(s)$ is the largest index $h \le i$ with $s_h \notin \{\sigma, ?\}$ if this exists (and 0 otherwise).

This polynomial time Dynamic Programming approach allows for efficient solving of most puzzles from newspapers. One can repeatedly apply the method to all rows or all columns (sweeps), thereby introducing a difficulty measure.

See K.J. Batenburg & WAK, Solving Nonograms by combining relaxations, Pattern Recognition 42 (2009) 1672–1683.

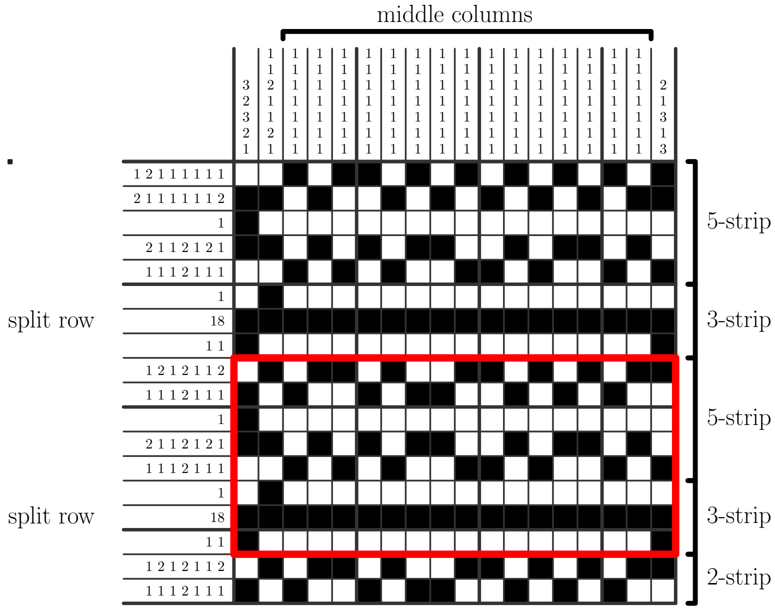


percentage unsolved pixels for randomly generated puzzles of different size & percentage **black**

So a difficulty measure could be: How many sweeps are needed to solve a given puzzle?

There exist $m \times n$ puzzles that require $\approx mn/2$ sweeps.

An 18×18 example, requiring 115 sweeps:



K.J. Batenburg, S. Henstra, WAK & W.J. Palenstijn, Constructing Simple Nonograms of Varying Difficulty, Pure Mathematics and Applications 20 (2009) 1–15 (*).

How far can we get by looking at a single row/column? Again, with • for a white pixel, and ● for a filled one:

	1	1 1	1	1 1	1
0					
1,1					
0					
1,1					
3					

•	•	•	•	•
•	●	•	●	•
•	•	•	•	•
?	?	•	?	?
?	?	●	?	?

But now we are stuck . . . unless we use rows and columns *together*.

We have this:

	1	1 1	1	1 1	1
0					
1,1					
0					
1,1					
3					

•	•	•	•	•
•	●	•	●	•
•	•	•	•	•
v	w	•	?	?
u	x	●	?	?

Suppose that $u =$ ●, then (column) v must be •, and so (row) $w =$ ●, and therefore (column) x must be •. Contradiction (row)! So u must be •.

The rest is simple.

The logic we used here has rules like “if this pixel is red, that pixel must be white”. This can be modeled through a 2-SAT problem, which happens to be solvable in polynomial time — in contrast with 3-SAT, which is NP-complete.

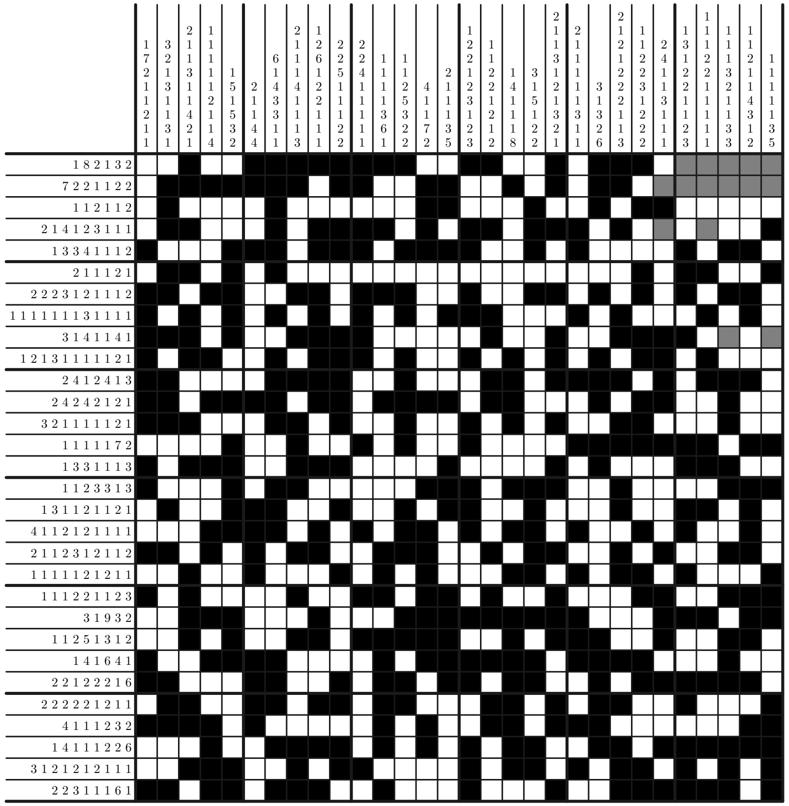
This offers another dimension for a difficulty measure.

Solving a Nonogram in general is NP-complete. In fact, Ueda and Nagao in 1996 even showed it to be “ASP-complete”: given a solution, it is NP-hard to determine if there is *another* solution. (This also holds for 3-SAT, but *not* for Graph 3-Coloring!)

As an illustration that this 2-SAT logic sometimes fails to catch everything:

	2	1	1 1	1	1
1	?		?		?
2	?	?	?	?	?
1	?		?		?
2	?	?	?	?	?
1	?		?		?

Partially solved 5×5 Nonogram, where the fact that pixel ○ must be white is hard(er) to infer.

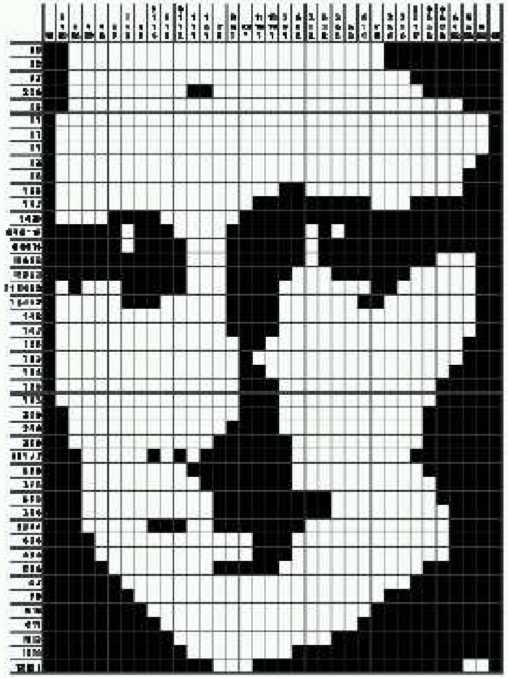


Randomly generated partially solved 30×30 Nonogram with 50 % **black** pixels; the grey cells denote the unknown pixels. This Nonogram has six solutions.

How to build = design your own Nonogram?

color picture

grey value picture



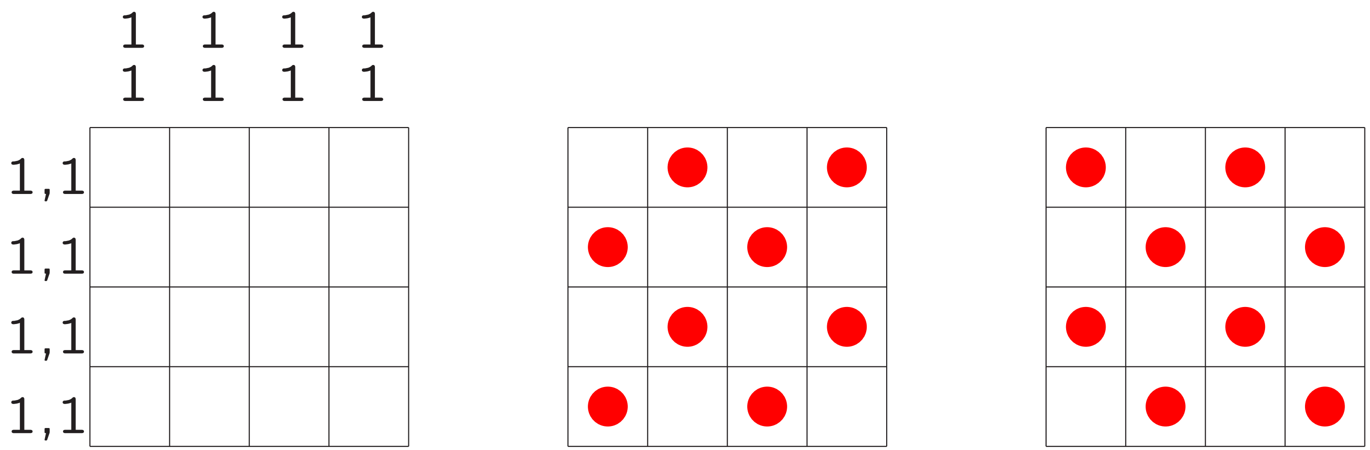
puzzle

See `www.liacs.nl/home/kosters/nono/` and (*) from slide 26.

Unique?

Remember that a *good* Nonogram should have a unique solution.

In general they have many different solutions with “some sort of” switching components!



We conclude: Discrete Tomography is an important (bio) research area, with many interesting algorithms and related complexity issues.

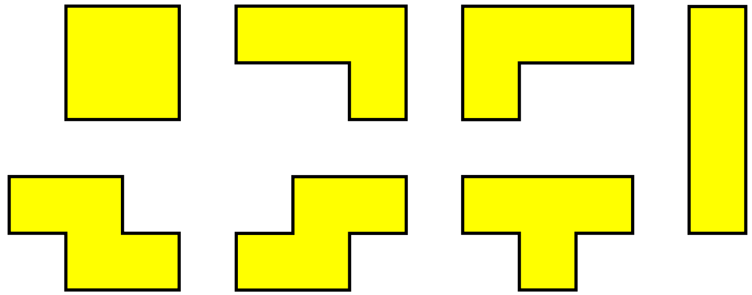
Vincent van Gogh

There are several interesting questions attached to Tetris:

- How to play well? (AI — Artificial Intelligence)
- How hard is it? (complexity: **IPA, July 8, 2011**)
- What might happen?

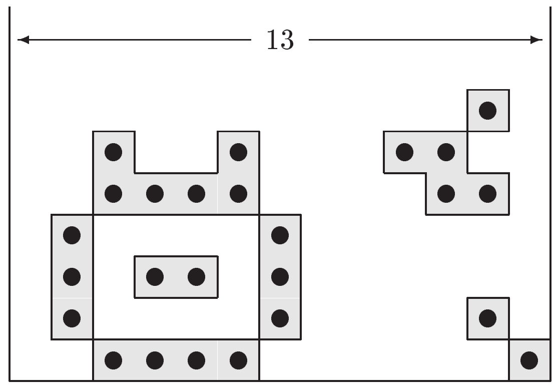
It has been shown that certain Tetris-problems are NP-complete (joint work with researchers from MIT & HJH), that you can reach almost all configurations, but that not all related problems are "decidable".

The 7 Tetris-pieces:



Random pieces fall down, and filled lines are cleared.
The question "Is it possible, given a finite ordered series of these pieces, to clear a partially filled game board?" is NP-complete.
If someone clears the board, this is easy to verify. If clearing is *not* possible however, up till now the only thing one can do to prove this is to check all possibilities, one by one!

An "arbitrary" configuration:



This figure can be made by dropping 276 suitable Tetris-pieces in the appropriate way, see

`www.liacs.nl/home/kosters/tetris/`

Claim: on a game board of odd width every configuration is reachable.