Structure (based on Lessons in Play, Chapter 6)

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April 27, 2020

Extremal games

- The greatest game born by day n is n.
- The least positive number born by day $n+1$ is 2^{-n}.
- The least positive game born by day $n+2$ is $⧾_n$.
- The maximal infinitesimals born by day $n+1$ are $n\times\uparrow$ and $n\times\uparrow *$.

Greatest game

Theorem: The greatest game born by day n is n.
Proof: Let G be any game born by day n. Then, its game tree is at most n deep, so any player can do at most n moves. Since left can do n moves in n, she can win $n - G$ by only playing in n. Therefore, $G \leq n$. Since this applies to any G, n has to be the greatest. ■

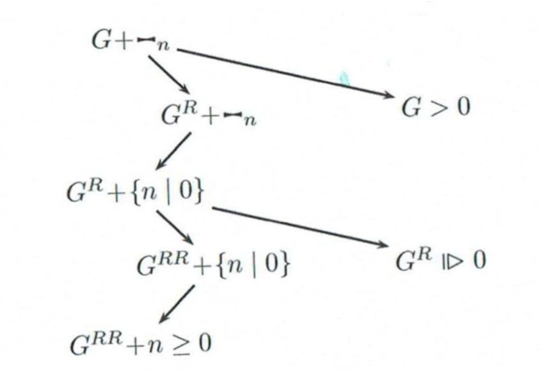
Least positive number

Theorem: The least positive number born by day $n+1$ is 2^{-n}.
Proof: For $n=0$, this gives $2^0=1$, which is indeed the least positive game born on day 1. For $n>0$, we can, without loss of generality, assume the least positive number born on day n to be in its canonical form, being in the form $\{y|z\}$. This is the smallest if $y=0$ and z is the smallest number born on day $n-1$, which by induction is 2^{1-n}. We get $\{0|2^{1-n}\}=2^{-n}$ ■

Least positive game

Theorem: The least positive game born by day $n+2$ is $\maltese_n$.
Proof: Let G be any positive game born by day n. In the game $G - \maltese_n$, Right going first can either move to G or to some $G^R - \maltese_n$. Left can win in G because $G > 0$ and in $G^R - \maltese_n$, Left can move to $G^R + \{n|0\}$. Again, Right hast two options: he can play to G^R or to $G^{RR} + \{n|0\}$. If Right plays to G^R, left has to have a winning move there because $G > 0$. If black moves to $G^{RR} + \{n|0\}$, Left can move to $G^{RR} + n$. G^{RR} is born on day n, so by Theorem 6.3, Left wins on $G^{RR} + n$. We can conclude that Left wins on $G - \maltese_n$ going second, so $G \leq \maltese_n$ Therefore, $\maltese_n$ is the smallest positive game born on day $n+2$. ■

Least positive game (cont.)



(strong) Number Avoidance

Theorem: If x is a number in canonical form with a left option and G is a game that's not a number, then there is a G^L such that $G^L + x > G + x^L$.

Number-Translation

Theorem: If X is a number and G is a game that's not a number, then $G + x = \{\mathcal{G}^L + x | \mathcal{G}^R + x\}$

Negative incentives

Theorem: If all of G's incentives are negative, then G is a number.

Cold, tepid and hot games

A game G is called:

- Cold if $\mathbf{LS}(G) < \mathbf{RS}(G)$. Then, G is a number.
- Tepid if $\mathbf{LS}(G) = \mathbf{RS}(G)$. Then, G is a number plus a non-zero infinitesimal.
- Hot if $\mathbf{LS}(G) > \mathbf{RS}(G)$. Games written as $\pm n$ are hot games.

Lattice

A lattice is a partial ordered set where for each pair of elements a and b, we have the following:

- Least upper bound/supremum/join, denoted $a \vee b$
- Greatest lower bound/infimum/meet, denoted $a \wedge b$

Lattice (cont.)

Theorem: The games born by day n form a lattice

Distributive lattice

A distributive lattice is a lattice in which the meet distributes over join, i.e. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge b)$. This is equivalent to join distributing over meet.

Distributive lattice(cont.)

Theorem: The games born by day n form a distributive lattice

Group structure

- As we discussed, games form a group.
- Linear combinations of a subset of group elements form a subgroup. We say the elements generate this subgroup.

Group structure day 0

- 1 element: 0.
- Generates the trivial group: only 0.

Group structure day 1

- 4 elements: 1, $*$, 0, -1.
- Independent generating set: $\{1, *\}$
- Generate a group ismorphic to $\mathbb{Z} \times \mathbb{Z}_2$

Group structure day 2

- 22 elements.
- Independent generating set: $\{\frac{1}{2}, *2, \{1|0\} - \{1|*\}\uparrow, \{1|0\} - \{1|0,*\}, \pm\frac{1}{2}, \pm 1\}$
- Generate a group ismorphic to $\mathbb{Z}^3 \times \mathbb{Z}_4 \times \mathbb{Z}_2^3$

Conclusion

We have previously structured games based on their birthday, but lots of games have the same birthday. We can now structure games further within a birthday.