Games Form a Group with a Partial Order (based on Lessons in Play, Chapter 4.2)

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Partial Order

A partial order is a binary relation ($\succeq$) with the following properties:

- Transitivity: if $x \succeq y$ and $y \succeq z$, then $x \succeq z$
- Reflexivity: for all x, $x \succeq x$
- Antisymmetry: if $x \succeq y$ and $y \succeq x$, then $x = y$

Partial Order (cont.)

Theorem: The relation $\geq$ is a partial order on games.

- Transitivity
- Reflexivity
- Antisymmetry

Partial Order (cont.)

Theorem: The relation $\geq$ is a partial order on games.

- Transitivity: given G, H and J such that $G \geq H$ and $H \geq J$, then Left can win playing second on $G - H$ and on $H - J$. By Lemma 3.3, left wins moving second on $(G - H) + (H - J)$, which, by Theorem 4.5 and Corollary 4.15 equals $G - J$, so $G \geq J$

Partial Order (cont.)

Theorem: The relation $\geq$ is a partial order on games.

- Reflexivity: by Corollary 4.15, $G - G = 0$, so by Theorem 4.12, it is a second player win, so $G \geq G$

Partial Order (cont.)

Theorem: The relation $\geq$ is a partial order on games.

- Antisymmetry: Exercise 4.20

Groups

A group is a set (S) equipped with a binary operation ($\bullet$) that satisfies the following properties:

- Closure: for all x and y in S, $x \bullet y \in S$
- Associativity: for all x, y and z in S, $(x \bullet y) \bullet z = x \bullet (y \bullet z)$
- Identity: there is a neutral element (e) in S such that for all x in S, $x \bullet e = e \bullet x = x$
- Inverse: For each x in S, there is an inverse element (x^{-1}) in S such that $x \bullet x^{-1} = x^{-1} \bullet x = e$

A group is called Abelian if the operation is commutative ($x \bullet y = y \bullet x$ for all x, y).

Groups (cont.)

Theorem: Games equipped with $+$ form an Abelian group.

- Closure: by definition
- Associativity: Theorem 4.5
- Identity: neutral element 0, by Theorem 4.4
- Inverse: inverse of x is $-x$, by Corollary 4.15
- Commutativity: Theorem 4.5

Partially ordered groups

A partially ordered group is a group whose elements form a partial ordering with the following extra property.

- Translation-invariance: If $x \succeq y$ then $x \bullet z \succeq y \bullet z$ and $z \bullet x \succeq z \bullet y$

Partially Ordered Groups

A partially ordered group is a group whose elements form a partial ordering with the following extra property.

- For games: If $G \geq H$ then $G + J \geq H + J$

Games have this property by Theorem 4.23.

Impartial Games

Impartial games are games where both players have the same options at all times, like Cram and Nim. They have a few special properties:

- Impartial games are their own inverse.
- Unequal impartial games are incomparable.

Impartial Games (cont.)

- Impartial games are their own inverse: given an impartial game G, $G + G$ allows either player playing second to win by copying the other player's moves, so $G + G \in P$. Now, by Theorem 4.12, $G + G = 0$. Because inverses in groups are unique, this implies that $G = -G$.

Impartial Games (cont.)

- Unequal impartial games are incomparable: given impartial games G and H such that $G \neq H$. Because of the possibility of strategy stealing, $G - H$ can't be in L or R, so it is in either P or N. Because $H \neq G = -G$ and inverses in groups are unique $G - H \neq 0$, so $G - H \notin P$. Therefore $G - H \in N$ and $G \parallel H$.

Order and Sums of 0, 1, -1 and $*$

Exercise: what games can we make by adding 0, 1, -1 and $*$, and how are they ordered?

Conclusion

- Games form a partially ordered group.
- This allows us to use the many theorem already proven for this structure.