Reduction to Canonical form

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Introduction

For every game G, there is a unique smallest (in terms of size and birthday) game that is equal to it. This game is G's canonical form. G is said to be in canonical form if:

1. G and all of its positions have no dominated options
2. G has no reversible options

Introduction

Thus, in order to reduce a game G to its unique canonical form:

- All dominated options must be removed
- All reversible options must be by-passed.

Recall: Theorem 4.33

Theorem 4.33

If $G = \{A, B, \dots \mid H, I, \dots\}$, then A and H are dominated left and right options, respectively, if $B \geq A$ and $I \leq H$, and $G' = \{B, \dots \mid I, \dots\}$, then $G = G'$.

Important Note: *G''s game tree is of smaller size than G's game tree since it can be obtained from G by pruning the subtree of which the dominated option is the root*

Recall: Theorem 4.34

Theorem 4.34

Suppose that some game $\boldsymbol{G} = \{\boldsymbol{A}, \boldsymbol{B}, \ldots \ |\boldsymbol{H}, \boldsymbol{I}, \ldots\}$ and $\boldsymbol{A}$ has some right option $\boldsymbol{A}^{\boldsymbol{R}}$ such that $\boldsymbol{G} \geq \boldsymbol{A}^{\boldsymbol{R}}$ with $\boldsymbol{A}^{\boldsymbol{R}}$'s left options given by $\{\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}, \ldots\}$.
Define $\boldsymbol{G}' = \{\boldsymbol{W}, \boldsymbol{X}, \boldsymbol{Y}, \ldots, \boldsymbol{B}, \boldsymbol{C}, \ldots|\boldsymbol{H}, \boldsymbol{I}, \boldsymbol{J}, \ldots\}$, and we have $\boldsymbol{G} = \boldsymbol{G}'$. The same case can be made for reversible right options.

Important Note: *$\boldsymbol{G}'$ is of smaller size than $\boldsymbol{G}$ since its game tree can be obtained by pruning the subtree of $\boldsymbol{G}$ of which $\boldsymbol{A}$ is the root and adding the left-options of $\boldsymbol{A}^{\boldsymbol{R}}$ as left-descendants (left-options) of $\boldsymbol{G}$*

Reduction Process

In summary, in order to obtain the canonical form of a game $\boldsymbol{G}$, one must remove all dominated options and bypass all reversible options. Note that this process is necessarily finite, since both sub-procedures (removal of dominated options and bypassing of reversible options) yield a new game that has a game tree that is smaller than $\boldsymbol{G}$'s game tree in terms of size or birthday or both.

Theorem 4.36

Theorem 4.36

If G and H are in canonical form and $G = H$, then $G \cong H$.

Some comments on Theorem 4.36:

- The theorem essentially states that G and H must have identical game trees
- The previous point implies that every game G has a unique canonical form, since it follows that every game that is equal to G and that is in canonical form must be isomorphic to G as well.

Proof of Theorem 4.36

Proof Theorem 4.36

If G and H are in canonical form and $G = H$, then $G \cong H$.

Proof:

1. $G = H \rightarrow G - H = 0 \rightarrow G - H$ is a P-position
2. Thus, Left can win moving second on $G - H$ and Left will have a winning response to $G^R - H$ for any right-option G^R of G
3. Suppose Left's winning response is in G^R, and we have $G^{RL} - H \geq 0 \rightarrow G^{RL} \geq H \rightarrow G^{RL} \geq G$.

Proof of Theorem 4.36 (Continued)

Proof Theorem 4.36 (Continued)

If G and H are in canonical form and $G = H$, then $G \cong H$.

Proof:

4. Point 3 implies that G has a reversible option which contradicts the fact that G is in canonical form. Hence, Left's winning response to $G^R - H$ must be in H
5. Since Left's winning response to $G^R - H$ must be in H, we find that $G^R - H^R \geq 0$, and thus $G^R \geq H^R$ for some H^R.
6. Point 1 and 5 shown above imply that for each right-option G^R of G for some right-option H^R of H, $G^R \geq H^R$ holds true.

Proof of Theorem 4.36 (Continued)

Proof Theorem 4.36 (Continued)

If G and H are in canonical form and $G = H$, then $G \cong H$.

Proof:

7. Repeating this argument, we can deduce that for each right-option H^R of H for some right-option $G^{R'}$ of G, $H^R \geq G^{R'}$ holds true.
8. From 7 it follows that for each G^R, $G^R \geq H^R \geq G^{R'}$ for some H^R and some G^R.
9. In point 8, G^R and $G^{R'}$ must be identical (otherwise at least one of G's right options would be dominated, which contradicts that G is in canonical form). Thus, $G^R = G^{R'}$ and $G^R = H^R = G^{R'}$

Proof of Theorem 4.36 (Continued)

Proof Theorem 4.36 (Continued)

If $\boldsymbol{G}$ and $\boldsymbol{H}$ are in canonical form and $\boldsymbol{G} = \boldsymbol{H}$, then $\boldsymbol{G} \cong \boldsymbol{H}$.

Proof:

10. It follows from point 9 that every right-option $\boldsymbol{G^R}$ of $\boldsymbol{G}$ matches up with some (one or multiple) right-option(s) $\boldsymbol{H^R}$ of $\boldsymbol{H}$. That is, $\mathcal{G}^R \subseteq \mathcal{H}^R$.
11. Repeating the previous argument (points 1 - 10), starting from the game $\boldsymbol{H} - \boldsymbol{G}$, Right playing first on $\boldsymbol{H}$, one can verify that $\mathcal{H}^R \subseteq \mathcal{G}^R$ holds true.

Proof of Theorem 4.36 (Continued)

Proof Theorem 4.36 (Continued)

If G and H are in canonical form and $G = H$, then $G \cong H$.

Proof:

12. From point 10 and 11 it follows that $\mathcal{H}^R = \mathcal{G}^R$
13. Repeating the previous argument (points 1 - 12), but with Left playing first, one can verify that we have that $\mathcal{H}^L \subseteq \mathcal{G}^L$ and $\mathcal{G}^L \subseteq \mathcal{H}^L$ which implies that $\mathcal{H}^L = \mathcal{G}^L$
14. We now have that $\mathcal{H}^L = \mathcal{G}^L$ and $\mathcal{H}^R = \mathcal{G}^R$ which implies that $G \cong H$.

Lemma 4.38

Lemma 4.38

If $G = H$ and G is in canonical form, then each option of G is dominated by an option of H; i.e,

1. $(\forall G^L)(\exists H^L)$ such that $H^L \geq G^L$, and
2. $(\forall G^R)(\exists H^R)$ such that $H^R \leq G^R$.

On the next two slides, separate proofs are given for both parts of the Lemma. (Part 1 and Part 2). Note that the proofs of both parts are similar to one another and similar to the proof of Theorem 4.36.

Proof Lemma 4.38

Proof Lemma 4.38 (Part 1)

1. $G = H \to G - H = 0 \to (G - H)$ is a P-position.
2. If Left plays first and plays to any left-option G^L of G, the resulting position will be $G^L - H$.
3. Right has a winning move on H (not on G^L, for then G^L would be a reversible option of G which contradicts the assumption that G is in canonical form)
4. Thus, Right can move to some H^L of H, such that $G^L - H^L \leq 0 \to G^L \leq H^L$

Proof Lemma 4.38

Proof Lemma 4.38 (Part 2)

1. $G = H \rightarrow G - H = 0 \rightarrow (G - H)$ is a P-position.
2. If Right plays first and plays to any right-option G^L of G, the resulting position will be $G^R - H$.
3. Left has a winning move on H (not on G^R, for then G^R would be a reversible option of G which contradicts the assumption that G is in canonical form)
4. Thus, Left can move to some H^R of H, such that $G^R - H^R \geq 0 \rightarrow G^R \geq H^R$

Exercise 4.40

Exercise 4.40

Suppose that $G = \mathbf{0}$ with $\mathcal{G}^L \neq \emptyset$, that is, G is not in canonical form. Show that if $G^L \in \mathcal{G}^L$, then G^L is reversible.

In light of Exercise 4.40, the following observations are worth mentioning:

- The canonical form of any game $G = \mathbf{0}$ is obviously the empty game, since it has the smallest possible game-tree.
- In order to prove the previous point, we can repeat the argument that proves that $\mathcal{G}^L \neq \emptyset$ to prove that $\mathcal{G}^R \neq \emptyset$.

Exercise 4.40 - Proof

Exercise 4.40 Proof

1. $\boldsymbol{G} = \mathbf{0} \rightarrow \boldsymbol{G}$ is a P-position.
2. Since $\boldsymbol{G}$ is a P-position, Right can win playing second on $\boldsymbol{G}$.
3. Should L play to some left-option $\boldsymbol{G^L}$ of $\boldsymbol{G}$, $\boldsymbol{G^L}$ must have at least one right-option, for otherwise Left could win playing first on $\boldsymbol{G}$ which contradicts that $\boldsymbol{G} = \mathbf{0}$.
4. Since Right can win playing second, there exists some option $\boldsymbol{G^{LR}}$ of $\boldsymbol{G^L}$, such that $\boldsymbol{G^{LR}} \leq \mathbf{0} = \boldsymbol{G}$ (equivalently, $\boldsymbol{G} \geq \boldsymbol{G^{LR}}$), which implies that $\boldsymbol{G^L}$ is a reversible option.

Example 4.41

Example 4.41

Find the canonical form of the $\mathbf{4} \times \mathbf{1}$ Domineering strip.

Solution:

Observe that the game is given by $\{\mathbf{1}, \mathbf{0}|\}$. Clearly, left-option $\mathbf{0}$ is dominated by left-option $\mathbf{1}$, hence the game can be simplified to $\{\mathbf{1}|\}$

Example 4.43

Example 4.43

Show that the canonical form of $G = \{-5 | -2\}$ is $G = -3$.
<u>**Solution:**</u>
G's right option -2 is not reversible, since it has no left-options. However, G's left-option -5 has right-option $G^{LR} = -4$ Note that $G - G^{LR} \geq 0$, and thus $G \geq G^{LR}$. Thus, -5 is a reversible option and -4 is the reversing option. -4 has no left-options, therefore the replacement set is empty, and we have that
$G = \{| -2\} = -3$

Example 4.46

Example 4.46

Show that the canonical form of $\{2, \{20| -10\}|1\}$ is $\{2|1\}$.

<u>**Solution:**</u>

The game is positive and the right option of $\{20| -10\}$ is negative, thus it reverses out and is replaced by the Left option of -10, which does not exist and so $\{20| -10\}$ disappears.

Conclusions

Conclusions

To conclude this presentation:

- For each set of games that are mutually equal, there exists a unique smallest version (e.g. the canonical form).
- Given any game G it is possible to obtain the canonical form by removing its dominated and reversible options, one by one.