Eigenspaces of the Laplace-Beltrami-operator on $SL(n,\mathbb{R})/S(GL(1)\times GL(n-1))$. Part I

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ABSTRACT

In this paper we consider the symmetric space mentioned in the title. We give a compactification of this space X. Then the eigenspaces of the Laplace-Beltrami-operator on X are determined. Loosely speaking these are given in terms of Poisson-transforms of certain hyperfunction-spaces on the boundary of X in its compactification. In order to show that the Poisson-transform is injective and surjective, an explicit inverse, the boundary value map, is constructed. Its existence relies upon the theory of Kashiwara and Oshima on regular singularities.

The first part of the paper deals with the compactification and the boundary value map.

In the second part the Poisson-transform is discussed.

1. INTRODUCTION

In this paper we consider the space $X=G/H$, where $G=SL(n,\mathbb{R})$ and $H=S(GL(1)\times GL(n-1))$, for $n\geqq 3$. X can be provided with a G-equivariant pseudo-Riemannian structure. The corresponding Laplace-Beltrami-operator on X is denoted by $\Box$. We are interested in the spaces V_λ, where

$$V_\lambda=\{f \text{ hyperfunction on } X;\ \Box f=\lambda f\}$$

for fixed complex λ. We shall construct the so-called Poisson-transform, which maps a certain function space B into V_λ. For the description of B we need a parabolic subgroup $\bar{P}$ of G, which is associated to H in a natural way. B consists of hyperfunctions on G which have a certain transformation property under right translation by elements of $\bar{P}$; this property will depend on λ.

The main subject of this paper is to show that the Poisson-transform is an isomorphism between the two spaces mentioned above. Therefore, we construct an explicit inverse: the boundary value map. The existence of this map relies heavily upon the theory of Kashiwara and Oshima on regular singularities [8].

The analogous problem for Riemannian symmetric spaces of the non compact type was solved by Kashiwara et al. [7], and for a large class of non Riemannian symmetric spaces by Oshima and Sekiguchi in [13]. In [15] Sekiguchi deals with this problem for several so-called rank one spaces, like $SO(p,q)/SO(p,q-1)$. However, the space X mentioned before is not treated there. In [11] and [12] Oshima claims that for general symmetric spaces the Poisson-transform is a G-equivariant isomorphism between the appropriate spaces. Nevertheless, no proof is given yet.

Let us give a brief outline of Oshima's approach, which we shall follow in this paper. First we give a compactification of X; this is used to define the boundary value map. Then the Poisson-transform is defined. At that time we can state the main theorem: see Chapter 5. In the last two chapters one can find the proofs of two key-lemmas. The main problem, especially in the last two chapters, is the following: in contrast with the situation in [15], the set $\hat{M}(\sigma)^i$ as defined in [11], consists of two elements. Therefore, we need to examine a certain non-trivial representation of $SO(n)$, which causes a lot of technical complications. As a consequence, we get two c-functions instead of one like in [15]. This also explains the fact that the space B splits up into two subspaces, corresponding to certain representations of $SO(n)$.

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2. DEFINITIONS AND PRELIMINARIES

Consider the connected real semisimple Lie group with finite centre $G=SL(n,\mathbb{R})$, the $n\times n$ real matrices with determinant one, for $n\geqq 3$. Define an involutive automorphism σ of G by $\sigma(g)=JgJ$ $(g\in G)$, where

$$J=\begin{pmatrix} -1 & & & \theta \\ & 1 & & \\ & & \ddots & \\ \theta & & & 1 \end{pmatrix}.$$

The set of fixed points of σ is easily computed: it is equal to $H=S(GL(1,\mathbb{R})\times \times GL(n-1,\mathbb{R}))$, where $GL(m,\mathbb{R})$ is the group of $m\times m$ real matrices with nonzero determinant. So we have:

$$H=\{g\in G\,|\,\sigma(g)=g\}=\left\{\begin{pmatrix} * & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix}\in G\right\}.$$

Note that H is not connected: it consists of two connected components. We let $X = G/H$; X is a semisimple symmetric space.

Besides σ, we also consider the involutive automorphism θ of G, defined by $\theta(g) = {}^t g^{-1}$ $(g \in G)$. θ is a so-called Cartan-involution and $\sigma\theta = \theta\sigma$. The set of fixed points of θ is equal to $K = SO(n, \mathbb{R})$, the $n \times n$ real orthogonal matrices. G/K is a Riemannian semisimple symmetric space of the non compact type.

Let $\mathfrak{g}$ be the Lie algebra of G: $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$, the $n \times n$ real matrices of trace zero. The differentials of σ and θ, again denoted by σ and θ, are involutive automorphisms of $\mathfrak{g}$. One has $\sigma(X) = JXJ$ and $\theta(X) = -{}^t X$, for $X \in \mathfrak{g}$. Computing the eigenspaces of σ and θ one obtains:

$$\mathfrak{h} = \{X \in \mathfrak{g} \,|\, \sigma(X) = X\} = \left\{ \begin{pmatrix} * & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & * & \\ 0 & & & \end{pmatrix} \in \mathfrak{g} \right\}$$

$$\mathfrak{q} = \{X \in \mathfrak{g} \,|\, \sigma(X) = -X\} = \left\{ \begin{pmatrix} 0 & * & \dots & * \\ * & & & \\ \vdots & & \theta & \\ * & & & \end{pmatrix} \in \mathfrak{g} \right\}$$

$$\mathfrak{k} = \{X \in \mathfrak{g} | \theta(X) = X\} = \{\text{skew symmetric matrices in } \mathfrak{g}\}$$

$$\mathfrak{p} = \{X \in \mathfrak{g} | \theta(X) = -X\} = \{\text{symmetric matrices in } \mathfrak{g}\}.$$

Now $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{k}$ that of K. We have:

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{k} \cap \mathfrak{q} \oplus \mathfrak{p} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q},$$

because σ and θ commute. In $\mathfrak{p} \cap \mathfrak{q}$ we choose a maximal abelian subspace $\mathfrak{a}$:

$$\mathfrak{a} = \left\{ \begin{pmatrix} & & t \\ & \theta & \\ t & & \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}.$$

Note that $\mathfrak{a}$ is also maximal abelian in $\mathfrak{q}$. Define $\alpha_0 \in \mathfrak{a}^*$, the real dual of $\mathfrak{a}$, by

$$\alpha_0 \begin{pmatrix} & & t \\ & \theta & \\ t & & \end{pmatrix} = t \quad (t \in \mathbb{R}).$$

We can diagonalize the action of ad $\mathfrak{a}$ on $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}(0) \oplus \mathfrak{g}(\alpha_0) \oplus \mathfrak{g}(2\alpha_0) \oplus \mathfrak{g}(-\alpha_0) \oplus \mathfrak{g}(-2\alpha_0),$$

where we defined $\mathfrak{g}(\alpha) = \{X \in \mathfrak{g} | [Y, X] = \alpha(Y)X \text{ for all } Y \in \mathfrak{a}\}$ $(\alpha \in \mathfrak{a}^*)$; $\mathfrak{g}(0)$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$. Let $\mathfrak{gl}(m, \mathbb{R})$ denote the set of $m \times m$ real matrices; then we have:

$$\mathfrak{g}(0)=\left\{\begin{pmatrix} t & 0 & \dots & 0 & s \\ 0 & & & & 0 \\ \vdots & & C & & \vdots \\ 0 & & & & 0 \\ s & 0 & \dots & 0 & t \end{pmatrix} \;\middle|\; s,t\in\mathbb{R};\ C\in\mathfrak{gl}(n-2,\mathbb{R});\ 2t+\operatorname{trace} C=0\right\}$$

$$\mathfrak{g}(\alpha_0)=\left\{\begin{pmatrix} 0 & x & 0 \\ y & \Theta & -y \\ 0 & x & 0 \end{pmatrix} \;\middle|\; x,y\in\mathbb{R}^{n-2}\right\}.$$

Here x is a row-vector and y a column-vector. This notation will be used throughout the paper.

$$\mathfrak{g}(2\alpha_0)=\left\{\begin{pmatrix} s & 0 & \dots & 0 & -s \\ 0 & & & & 0 \\ \vdots & & \Theta & & \vdots \\ 0 & & & & 0 \\ s & 0 & \dots & 0 & -s \end{pmatrix} \;\middle|\; s\in\mathbb{R}\right\}$$

$$\mathfrak{g}(-\alpha_0)=\sigma(\mathfrak{g}(\alpha_0))=\theta(\mathfrak{g}(\alpha_0))=\left\{\begin{pmatrix} 0 & x & 0 \\ y & \Theta & y \\ 0 & -x & 0 \end{pmatrix} \;\middle|\; x,y\in\mathbb{R}^{n-2}\right\}$$

$$\mathfrak{g}(-2\alpha_0)=\sigma(\mathfrak{g}(2\alpha_0))=\theta(\mathfrak{g}(2\alpha_0))=\left\{\begin{pmatrix} s & 0 & \dots & 0 & s \\ 0 & & & & 0 \\ \vdots & & \Theta & & \vdots \\ 0 & & & & 0 \\ -s & 0 & \dots & 0 & -s \end{pmatrix} \;\middle|\; s\in\mathbb{R}\right\}.$$

Note that $\mathfrak{g}(0)=\mathfrak{a}\oplus\mathfrak{m}$, where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{h}$. In this way we get a root system of type $(BC)_1$, namely $\{\pm\alpha_0, \pm 2\alpha_0\}$. Define $\mathfrak{n}=\mathfrak{g}(\alpha_0)\oplus\mathfrak{g}(2\alpha_0)$ and $\bar{\mathfrak{n}}=\mathfrak{g}(-\alpha_0)\oplus\mathfrak{g}(-2\alpha_0)$; both are nilpotent Lie algebras. We give a table of the dimensions:

'Lie algebra'	$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{q}$	$\mathfrak{k}$	$\mathfrak{p}$	$\mathfrak{a}$	$\mathfrak{p}\cap\mathfrak{q}$	$\mathfrak{m}$	$\mathfrak{n}$
Dimension	n^2-1	$(n-1)^2$	$2(n-1)$	$\frac{n(n-1)}{2}$	$\frac{n(n+1)}{2}-1$	1	$n-1$	$(n-2)^2$	$2n-3$

Because $\mathfrak{a}$ is one-dimensional, $(\mathfrak{g},\mathfrak{h})$ is called a rank one symmetric pair. For any Lie subalgebra $\mathfrak{l}$ of $\mathfrak{g}$, we denote its complexification by $\mathfrak{l}_{\mathbb{C}}$. The Killing-form B of $\mathfrak{g}$ is $B(X,Y)=2n$ trace (XY) for $X,Y\in\mathfrak{g}$.

Consider the connected Lie subgroups A, N and $\bar{N}$ of G defined by $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$ and $\bar{N} = \exp \bar{\mathfrak{n}}$. Then:

$$A = \left\{ a_t = \begin{pmatrix} \operatorname{ch} t & & & & \operatorname{sh} t \\ & 1 & & \theta & \\ & & \ddots & & \\ & \theta & & 1 & \\ \operatorname{sh} t & & & & \operatorname{ch} t \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

$$N = \left\{ n(x,y,z) = \begin{pmatrix} 1+z & & x & & -z \\ & 1 & & \theta & \\ y & & \ddots & & -y \\ & \theta & & 1 & \\ z & & x & & 1-z \end{pmatrix} \;\middle|\; x,y \in \mathbb{R}^{n-2};\ z \in \mathbb{R} \right\}$$

$$\bar{N} = \left\{ \bar{n}(x,y,z) = \begin{pmatrix} 1+z & & x & & z \\ & 1 & & \theta & \\ y & & \ddots & & y \\ & \theta & & 1 & \\ -z & & -x & & 1-z \end{pmatrix} \;\middle|\; x,y \in \mathbb{R}^{n-2};\ z \in \mathbb{R} \right\}.$$

Note, that

$$\exp \begin{pmatrix} z & x & -z \\ y & \theta & -y \\ z & x & -z \end{pmatrix} = n(x, y, z + (x,y)),$$

where $(\cdot,\cdot)$ is the standard inner product on $\mathbb{R}^{n-2}$. One computes:

$$a_t a_s = a_{t+s};\ a_t n(x,y,z) a_{-t} = n(e^t x, e^t y, e^{2t} z)$$

$$a_t \bar{n}(x,y,z) a_{-t} = \bar{n}(e^{-t} x, e^{-t} y, e^{-2t} z)$$

$$n(x,y,z) n(x',y',z') = n(x+x', y+y', z+z'+(x,y')).$$

In this last equation one can replace n by $\bar{n}$. Now it is clear that N and $\bar{N}$ are Heisenberg groups. Define the Lie subgroup M of G by $M = \{h \in H \mid \operatorname{Ad}(h)Y = Y$ for all $Y \in \mathfrak{a}\}$. Then:

$$M = \left\{ m(\alpha, B) = \begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 0 & & & \vdots \\ \vdots & & B & 0 \\ 0 & \cdots & 0 & \alpha \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R};\ B \in GL(n-2, \mathbb{R});\ \alpha^2 \det B = 1 \right\}$$

M has two connected components and its Lie algebra is $\mathfrak{m}$. One verifies:

$$m(\alpha, B) n(x,y,z) m(\alpha, B)^{-1} = n\left(\alpha x B^{-1}, \frac{1}{\alpha} By, z\right).$$

Define the element w of K by

$$w=\begin{pmatrix} 0 & & & & & 1 \\ & -1 & & & & \\ & & 1 & & \Theta & \\ & & & \ddots & & \\ & & \Theta & & 1 & \\ 1 & & & & & 0 \end{pmatrix}.$$

Then $\mathrm{Ad}(w)Y=Y$ for all $Y\in\mathfrak{a}$; however $w\notin H$. Using w we construct a parabolic subgroup $\bar{P}$ of G, i.e. a Lie subgroup which has a conjugate containing the upper triangular matrices in G (these elements constitute a minimal parabolic subgroup of G). Let $\bar{M}=M\cup wM$, a Lie subgroup of G. Then $\bar{P}=\bar{M}AN$ is a parabolic subgroup of G as we shall see later on. We also need $P=MAN$, which is not parabolic. This is in sharp contrast with the spaces treated in [Sekiguchi, 15]. There the subgroup P, constructed in the same manner, is already parabolic. As mentioned in the introduction this difference causes a lot of trouble. Furthermore, the reader should be aware of the fact that $\bar{M}AN$ is not a Langlands decomposition of $\bar{P}$.

For some of our computations it is somewhat easier to use the so-called diagonal-form. Define $C\in K$ by

$$C=\begin{pmatrix} 1/\sqrt{2} & 0 & \dots & 0 & 1/\sqrt{2} \\ 0 & 1 & & \Theta & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \Theta & & 1 & 0 \\ -1/\sqrt{2} & 0 & \dots & 0 & 1/\sqrt{2} \end{pmatrix}$$

and consider the automorphism $g\mapsto CgC^{-1}$ of G. This is the Cayley-transform; its differential is of course an automorphism of $\mathfrak{g}$. Note, that

$$C\begin{pmatrix} & & 1 \\ & \Theta & \\ 1 & & \end{pmatrix}C^{-1}=\begin{pmatrix} 1 & & \\ & \Theta & \\ & & -1 \end{pmatrix}.$$

We shall denote $C\mathfrak{a}C^{-1}$ by $\mathfrak{a}'$, etc. In this way we can consider $\mathfrak{a}'$ as a subspace of the standard maximal abelian subspace $\mathfrak{a}'_{\mathfrak{p}}$ of $\mathfrak{p}=\mathfrak{p}'$, which consists of the diagonal matrices of trace zero; this is the reason why we used the word diagonal-form.

Now we do some computations for the Riemannian symmetric pair $(\mathfrak{g},\mathfrak{k})$; we use the diagonal-form. Let us agree to omit the primes, for simplicity. We get a root space decomposition

$$\mathfrak{g}=\mathfrak{a}_{\mathfrak{p}}\oplus\sum_{\alpha\in\Sigma}^{\oplus}\mathfrak{g}_{\alpha},$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [Y, X] = \alpha(Y)X \text{ for all } Y \in \mathfrak{a}_\mathfrak{p}\}$ ($\alpha \in \mathfrak{a}_\mathfrak{p}^*$, the real dual of $\mathfrak{a}_\mathfrak{p}$) and Σ is a subset of $\mathfrak{a}_\mathfrak{p}^*$: a root system of type A_{n-1}. For $SL(n, \mathbb{R})$ the root spaces are easily described. For fixed i, j in $\{1, 2, \ldots, n\}$ define E_{ij} to be the $n \times n$ matrix with only one nonzero element: the (i, j)-entry, which is equal to one. Define $\alpha_{ij} \in \mathfrak{a}_\mathfrak{p}^*$ by

$$\alpha_{ij} \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} = a_i - a_j \qquad (a_l \in \mathbb{R}).$$

Then $[Y, E_{ij}] = \alpha_{ij}(Y)E_{ij}$ ($Y \in \mathfrak{a}_\mathfrak{p}; i, j \in \{1, 2, \ldots, n\}$). So we have $\mathfrak{g}_{\alpha_{ij}} = \{tE_{ij} \mid t \in \mathbb{R}\}$ for $i \neq j$, and $\Sigma = \{\alpha_{ij} \mid i, j = 1, 2, \ldots, n; i \neq j\}$. Call α_{ij} positive if $i < j$; this gives an ordering of the root system Σ. Define

$$\mathfrak{n}_{\min} = \sum_{\alpha \in \Sigma, \alpha > 0}^{\oplus} \mathfrak{g}_\alpha.$$

If we define $\alpha_0 \in \mathfrak{a}^*$ to be positive, we get compatible orderings of the two root systems involved, i.e. if $\alpha \in \mathfrak{a}_\mathfrak{p}^*$, $\alpha > 0$, $\alpha|_\mathfrak{a} \neq 0$ then $\alpha|_\mathfrak{a} > 0$. This is clear from the fact that $\alpha_{1j}|_\mathfrak{a} = \alpha_{jn}|_\mathfrak{a} = \alpha_0$ ($j \in \{2, \ldots, n-1\}$) and $\alpha_{1n}| = 2\alpha_0$. Note that

$$C\mathfrak{n}C^{-1} = \left\{ \begin{pmatrix} 0 & x & s \\ & & y \\ 0 & & 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R}^{n-2};\ s \in \mathbb{R} \right\}.$$

All root spaces are one-dimensional.

Define some special elements of $\mathfrak{a}_{\mathfrak{p},\mathbb{C}}^*$, the space of $\mathbb{C}$-linear maps from $\mathfrak{a}_{\mathfrak{p},\mathbb{C}}$ to $\mathbb{C}$, by:

$$\varrho_\mathfrak{p}(Y) = \tfrac{1}{2} \operatorname{trace}(\operatorname{ad} Y|\mathfrak{n}_{\min,\mathbb{C}}) = \tfrac{1}{2} \sum_{\alpha > 0} \alpha(Y) \quad (Y \in \mathfrak{a}_{\mathfrak{p},\mathbb{C}}^*)$$

$$\varrho_1 = \frac{n-1}{2}\, \alpha_{1n}$$

$$\lambda(s) = \varrho_\mathfrak{p} + \left(\frac{s}{n-1} - 1\right)\varrho_1 \quad (s \in \mathbb{C}).$$

Note that $\lambda(n-1) = \varrho_\mathfrak{p}$ and $\varrho_1|_\mathfrak{a} = (n-1)\alpha_0$, which is the analogon of $\varrho_\mathfrak{p}$ for $\mathfrak{a}$. For an element $\alpha \in \mathfrak{a}_{\mathfrak{p},\mathbb{C}}^*$ we define a unique element H_α of $\mathfrak{a}_{\mathfrak{p},\mathbb{C}}$ by $B(H_\alpha, Y) = \alpha(Y)$ for all $Y \in \mathfrak{a}_{\mathfrak{p},\mathbb{C}}$. Then we can transfer the Killing-form from $\mathfrak{a}_{\mathfrak{p},\mathbb{C}}$ to $\mathfrak{a}_{\mathfrak{p},\mathbb{C}}^*$: for $\alpha, \beta \in \mathfrak{a}_{\mathfrak{p},\mathbb{C}}^*$, let $\langle \alpha, \beta \rangle = B(H_\alpha, H_\beta)$. For $\lambda \in \mathfrak{a}_{\mathfrak{p},\mathbb{C}}^*$, define:

$$I(\lambda) = \prod_{\substack{\alpha > 0 \\ \alpha \in \Sigma}} \Gamma(\tfrac{1}{2})\Gamma\left(\frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\right) \Big/ \Gamma\left(\tfrac{1}{2} + \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\right)$$

$$c(\lambda) = I(\lambda)/I(\varrho_\mathfrak{p})$$

$$e(\lambda)^{-1} = \prod_{\substack{\alpha > 0 \\ \alpha \in \Sigma}} \Gamma\left(\tfrac{1}{2}\left(\tfrac{3}{2} + \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\right)\right)\Gamma\left(\tfrac{1}{2}\left(\tfrac{1}{2} + \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\right)\right).$$

Here $c(\lambda)$ is Harish-Chandra's c-function, and $e(\lambda)^{-1}$ is its denominator. We have:

LEMMA 2.1. $s \in \mathbb{C}$. *Then*

$$c(\lambda(s)) = \frac{(n-2)!2^{2-n}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s-n+3}{4}\right)^2}{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+n-1}{4}\right)^2}.$$

PROOF.

$$I(\lambda) = \prod_{1 \leq i < j \leq n} B\left(\tfrac{1}{2}, \frac{\langle \lambda, \alpha_{ij} \rangle}{\langle \alpha_{ij}, \alpha_{ij} \rangle}\right).$$

We compute $\langle \lambda, \alpha_{ij} \rangle$ and $\langle \alpha_{ij}, \alpha_{ij} \rangle$. First note that

$$H_{\alpha_{ij}} = \frac{1}{2n}(E_{ii} - E_{jj}), \text{ so } \langle \alpha_{ij}, \alpha_{ij} \rangle = \frac{1}{n}.$$

Furthermore

$$\varrho_{\mathfrak{p}}(Y) = \tfrac{1}{2} \sum_{i=1}^{n} (n+1-2i)h_i,$$

for

$$Y = \begin{pmatrix} h_1 & & \Theta \\ & \ddots & \\ \Theta & & h_n \end{pmatrix} \text{ in } \mathfrak{a}_{\mathfrak{p},\mathbb{C}}.$$

Define $\gamma_i \in \mathfrak{a}^*_{\mathfrak{p},\mathbb{C}}$ by

$$\gamma_i \begin{pmatrix} h_1 & & \Theta \\ & \ddots & \\ \Theta & & h_n \end{pmatrix} = h_i.$$

Then we have

$$H_{\gamma_i} = \frac{1}{2n} E_{ii} - \frac{1}{2n^2} \sum_{j=1}^{n} E_{ii}.$$

Straightforward computation yields, with $i \neq j$,

$$\frac{\langle \lambda(s), \alpha_{ij} \rangle}{\langle \alpha_{ij}, \alpha_{ij} \rangle} = \begin{cases} \dfrac{j-i}{2} & \text{if } i \neq 1, j \neq n \\[2ex] \dfrac{s}{2} & \text{if } i = 1, j = n \\[2ex] \dfrac{s-n-1+2j}{4} & \text{if } i = 1, j \neq n \\[2ex] \dfrac{s+n+1-2i}{4} & \text{if } i \neq 1, j = n. \end{cases}$$

This implies, with nonzero constants c_1 and c_2, independent of s:

$$I(\lambda(s)) = c_1 B\left(\tfrac{1}{2}, \frac{s}{2}\right) \prod_{i=2}^{n-1} B\left(\tfrac{1}{2}, \frac{s+n+1-2i}{4}\right)^2 =$$

$$= c_2 \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \frac{\Gamma\left(\frac{s-n+3}{4}\right)^2}{\Gamma\left(\frac{s+n-1}{4}\right)^2}.$$

Using $\varrho_{\mathfrak{p}} = \lambda(n-1)$ and the duplication formula for the Γ-function we derive:

$$I(\varrho_{\mathfrak{p}}) = c_2 \frac{\sqrt{\pi}\, 2^{n-2}}{(n-2)!}.$$

Now the proof is easily completed. ∎

LEMMA 2.2. *$s \in \mathbb{C}$, $s \notin \mathbb{Z}$. Then $e(\lambda(s)) \neq 0$.*

PROOF.

$$e(\lambda)^{-1} = \prod_{\alpha>0} \left\{ \sqrt{2\pi}\, 2^{-(\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle)} \Gamma\left(\tfrac{1}{2} + \frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}\right)\right\},$$

by the duplication formula. Using the computations from the previous proof and the fact that $1/\Gamma(z)$ has zeros exactly in $\{0, -1, -2, \dots\}$ one derives: if $s \notin \{-1, -3, \dots\} \cup \{n-5, n-7, \dots\}$, then $e(\lambda(s)) \neq 0$, so in fact a better result is proved. ∎

Now we return to the minimal parabolic subgroup. We define $A_{\min} = \exp \mathfrak{a}_{\mathfrak{p}}$, $N_{\min} = \exp \mathfrak{n}_{\min}$ and $M_{\min} = \{k \in K \mid \mathrm{Ad}(k)Y = Y \text{ for all } Y \in \mathfrak{a}_{\mathfrak{p}}\}$. Then $P_{\min} = M_{\min}A_{\min}N_{\min}$ is a minimal parabolic subgroup, by definition. Note that

$$A'_{\min} = \left\{ \begin{pmatrix} e^{t_1} & & 0 \\ & \ddots & \\ 0 & & e^{t_n} \end{pmatrix} \,\middle|\, t_i \in \mathbb{R};\ \sum_{i=1}^{n} t_i = 0 \right\}$$

$$N'_{\min} = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in G \right\}$$

$$M'_{\min} = \left\{ \begin{pmatrix} \varepsilon_1 & & 0 \\ & \ddots & \\ 0 & & \varepsilon_n \end{pmatrix} \,\middle|\, \varepsilon_i = \pm 1;\ \prod_{i=1}^{n} \varepsilon_i = 1 \right\}.$$

Consider the following function spaces on a real analytic manifold Ω: $C(\Omega)$, $C^\infty(\Omega)$, $C_c^\infty(\Omega) = D(\Omega)$, $A(\Omega)$, $D'(\Omega)$ and $B(\Omega)$ are the spaces of continuous functions, C^∞ functions, C^∞ functions with compact support, real analytic functions, distributions and hyperfunctions, resp. For more information about hyperfunctions, the reader is referred to [Schlichtkrull, 14] and the references mentioned there.

Let $\mathbb{D}(G/K)$ be the space of G-invariant differential operators on G/K, i.e. those differential operators on G/K that commute with the left action of G on the homogeneous space G/K. We have the same notion for G/H: $\mathbb{D}(G/H)$, the space of G-invariant differential operators on G/H. Let us describe this space. Therefore we introduce the Laplace-Beltrami-operator $\square$ of G/H: identify $\mathfrak{q}$ with the tangent space to G/H at eH and use the restriction of the Killing-form to $\mathfrak{q}$ to give a pseudo-Riemannian structure on G/H. Standard differential geometry provides us with a G-invariant differential operator $\square$ on G/H, the so-called Laplace-Beltrami-operator. It is also possible to construct $\square$, or a nonzero multiple of it, by using the Casimir-element from the centre of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}_{\mathbb{C}}$. This construction will be given in Chapter 4. It is well-known that $\mathbb{D}(G/H)=\mathbb{C}[\square]$, so every G-invariant differential operator on G/H is a polynomial in $\square$.

Throughout this paper dk will be the normalized Haar measure of K: $\int_K dk=1$. Let log denote the inverse of the map exp: $\mathfrak{a}_{\mathfrak{p}}\to A_{\min}$. For $\lambda\in\mathfrak{a}^*_{\mathfrak{p},\mathbb{C}}$ we denote by χ_λ the corresponding homomorphism of $\mathbb{D}(G/K)$ into $\mathbb{C}$, see [Kashiwara et al., 7]. Now define, for $\lambda\in\mathfrak{a}^*_{\mathfrak{p},\mathbb{C}}$:

$$B(G/P_{\min};L_\lambda)=\{f\in B(G)\,|\,f(gman)=e^{(\lambda-\varrho_{\mathfrak{p}})(\log a)}f(g)$$
$$\text{for all } g\in G,\ m\in M_{\min},\ a\in A_{\min},\ n\in N_{\min}\}$$

$$A(G/K;M(\chi_\lambda))=\{f\in A(G/K)\,|\,Df=\chi_\lambda(D)f \text{ for all } D\in\mathbb{D}(G/K)\}$$

$$(\mathscr{P}_\lambda f)(g)=\int_K f(gk)dk\quad (f\in B(G/P_{\min};L_\lambda), g\in G)$$

$\mathscr{P}_\lambda$ is called the Poisson-transform. Note that both $B(G/P_{\min};L_\lambda)$ and $A(G/K;M(\chi_\lambda))$ are representation spaces for G:

$$(\pi_\lambda(g)f)(x)=f(g^{-1}x)\quad (g,x\in G;f\in B(G/P_{\min};L_\lambda))$$

$$(\tilde{\pi}_\lambda(g)f)(x)=f(g^{-1}x)\quad (g\in G;x\in G/K;f\in A(G/K;M(\chi_\lambda))).$$

That this last definition is a good one, follows from the invariance of the differential operators involved. Now the main result in [Kashiwara et al., 7] is:

THEOREM 2.3. *If* $\lambda\in\mathfrak{a}^*_{\mathfrak{p},\mathbb{C}}$ *satisfies* $e(\lambda)\neq 0$, *then* $\mathscr{P}_\lambda$ *is a G-equivariant isomorphism of* $B(G/P_{\min};L_\lambda)$ *onto* $A(G/K;M(\chi_\lambda))$, *so* $\mathscr{P}_\lambda\cdot\pi_\lambda=\tilde{\pi}_\lambda\cdot\mathscr{P}_\lambda$. ∎

In Chapter 5 we shall formulate an analogous theorem for the space G/H. Therefore, we need similar function spaces as considered above. Write $\varrho=n-1$; define for $s\in\mathbb{C}$:

$$B(G/P;s)=\{f\in B(G)\,|\,f(gma_tn)=e^{(s-\varrho)t}f(g)$$
$$\text{for all } g\in G,\ m\in M,\ t\in\mathbb{R},\ n\in N\}.$$

Noting that for $f \in B(G/P;s)$ we have a decomposition

$$f(g) = \tfrac{1}{2}(f(g) + f(gw)) + \tfrac{1}{2}(f(g) - f(gw))$$

it is clear that we get $B(G/P;s) = B_1(G/P;s) \oplus B_2(G/P;s)$, where

$$B_1(G/P;s) = \{f \in B(G/P;s) \mid f(gw) = f(g) \text{ for all } g \in G\},$$

$$B_2(G/P;s) = \{f \in B(G/P;s) \mid f(gw) = -f(g) \text{ for all } g \in G\}.$$

LEMMA 2.4. *Let* $s \in \mathbb{C}$. *Then* $B_1(G/P;s) \subset B(G/P_{\min}; L_{\lambda(s)})$.

PROOF. In the diagonal-form, writing A' instead of CAC^{-1}, etc., we have $M' = M$,

$$w' = \begin{pmatrix} 1 & & & & \theta \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 & \\ \theta & & & & & -1 \end{pmatrix}$$

$$A' = \left\{ \begin{pmatrix} e^t & & & & \theta \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \theta & & & & e^{-t} \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

$$N' = \left\{ \begin{pmatrix} 1 & x & s \\ & \begin{matrix} 1 & & \theta \\ & \ddots & \\ \theta & & 1 \end{matrix} & y \\ & & 1 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R}^{n-2};\ s \in \mathbb{R} \right\}$$

$$Ca_tC^{-1} = \begin{pmatrix} e^t & & \\ & \begin{matrix} 1 & & \theta \\ & \ddots & \\ \theta & & 1 \end{matrix} & \\ & & e^{-t} \end{pmatrix}.$$

Then $M'_{\min} \subset (M' \cup w'M')$, $A'_{\min} \subset M'A'$ and $N'_{\min} \subset M'N'$. This last statement is proved in the following way: take $n \in N'_{\min}$; then with appropriate $u, v \in \mathbb{R}^{n-2}$, $t \in \mathbb{R}$ and $(n-2) \times (n-2)$ matrix

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ \theta & & 1 \end{pmatrix},$$

we have

$$n=\begin{pmatrix}1 & u & t\\ & 1 \ddots * & \\ & \ddots & v\\ & 0 \quad 1 & \\ & & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & \dots & 0\\ & 1 \ddots * & & \vdots\\ & & \ddots & \\ & 0 & 1 & 0\\ & & & 1\end{pmatrix}\begin{pmatrix}1 & u & t\\ & 1 \ddots 0 & \\ & \ddots & v'\\ & 0 \quad 1 & \\ & & 1\end{pmatrix}\in M'N'$$

where

$$v'=\begin{pmatrix}1 & & *\\ & \ddots & \\ 0 & & 1\end{pmatrix}^{-1} v.$$

Let $f\in B_1(G/P;s)$, $g\in G$, $m\in M'_{\min}$, $a\in A'_{\min}$, $n\in N'_{\min}$. Then $f(gman)=f(ga)=$ $=e^{(\lambda(s)-\varrho_{\mathfrak{p}})(\log a)}f(g)$, using

$$(\lambda(s)-\varrho_{\mathfrak{p}})\left(\log\begin{pmatrix}e^{t_1} & & 0\\ & \ddots & \\ 0 & & e^{t_n}\end{pmatrix}\right)=(s-\varrho)\,\frac{t_1-t_n}{2}. \qquad \blacksquare$$

In order to give a similar inclusion for $B_2(G/P;s)$, we need the character χ of $M'_{\min}$, defined by

$$\chi\begin{pmatrix}\varepsilon_1 & & 0\\ & \ddots & \\ 0 & & \varepsilon_n\end{pmatrix}=\varepsilon_1\varepsilon_n,\quad \begin{pmatrix}\varepsilon_1 & & 0\\ & \ddots & \\ 0 & & \varepsilon_n\end{pmatrix}\in M'_{\min}.$$

Let

$$B(G/P_{\min};L_\lambda;\chi)=\{f\in B(G)\,|\,f(gman)=\chi(m)e^{(\lambda-\varrho_{\mathfrak{p}})(\log a)}f(g)$$
$$\text{for all } g\in G,\ m\in M_{\min},\ a\in A_{\min},\ n\in N_{\min}\}.$$

LEMMA 2.5. *Let $s\in\mathbb{C}$. Then $B_2(G/P;s)\subset B(G/P_{\min};L_{\lambda(s)};\chi)$.*

PROOF. We only have to remark that for $f\in B_2(G/P;s)$:

$$f(gw)=-f(g)=\chi(w)f(g).$$

The rest of the proof is an adaption of that of the previous lemma. $\blacksquare$

It is clear, that during the proof of Lemma 2.4 we also showed that $\bar{P}$ is a parabolic subgroup of G, because $P_{\min}\subset\bar{P}$. For $n=3$ one sees, that $\bar{P}=P_{\min}$. For $n>3$ however $\bar{P}$ is neither minimal nor maximal. For example, $\bar{P}'$ is contained in the parabolic subgroup

$$\left\{\begin{pmatrix} & * & \\ 0 & \dots & 0\,*\end{pmatrix}\in G\right\}$$

of G, which is strictly larger than $\bar{P}'$.

Now we introduce some notations from [Kosters, 9], Chapter 4. Define the following $n\times n$ matrices:

$$x^0=\begin{pmatrix}1 & \\ & \theta\end{pmatrix};\ \xi^0=\begin{pmatrix}1 & & -1\\ & \theta & \\ 1 & & -1\end{pmatrix};\ \overline{\xi^0}=\begin{pmatrix}1 & & 1\\ & \theta & \\ -1 & & -1\end{pmatrix}.$$

For $x,y\in \mathfrak{gl}(n,\mathbb{R})$, let $P(x,y)=\text{trace }(xy)$. G acts on $\mathfrak{gl}(n,\mathbb{R})$ in the obvious way: $g\cdot x=gxg^{-1}$ $(g\in G,\ x\in\mathfrak{gl}(n,\mathbb{R}))$. Note that

$$H=\{g\in G\,|\,g\cdot x^0=x^0\}$$
$$MN=\{g\in G\,|\,g\cdot \xi^0=\xi^0\}$$
$$M\bar{N}=\{g\in G\,|\,g\cdot \overline{\xi^0}=\overline{\xi^0}\}.$$

The orbit $G\cdot x^0$ is equal to

$$\{x\in\mathfrak{gl}(n,\mathbb{R})\,|\,\text{rank }x=\text{trace }x=1\}$$

and can be identified with G/H. Define $P_0(x)=P(x,\xi^0)$, for $x\in G/H$. Note that

$$P_0(g\cdot x^0)=(g_{11}-g_{n1})(g_{11}^{-1}+g_{1n}^{-1})\quad (\text{let } g_{11}^{-1}=(g^{-1})_{11},\text{ etc.}).$$

We define the Poisson-kernels by

$$P_s^1(g)=|P(g\cdot x^0,\xi^0)|^{(-s-\varrho)/2}$$
$$P_s^2(g)=|P(g\cdot x^0,\xi^0)|^{(-s-\varrho)/2}\,sgn(P(g\cdot x^0,\xi^0))$$

for $s\in\mathbb{C}$, $g\in G$. Then we have $P_s^1(e)=P_s^2(e)=1$;

$$P_s^i(ma_tngh)=e^{(s+\varrho)t}P_s^i(g)\ (i=1,2;\ g\in G,\ m\in M,\ t\in\mathbb{R},\ n\in N,\ h\in H);$$
$$P_s^1(wg)=P_s^1(g);\ P_s^2(wg)=-P_s^2(g)\ (g\in G).$$

We need several decompositions of G. First we have the generalized Cartan-decomposition. Let

$$C^\infty(K\setminus G/H)=\{f\in C^\infty(G)\,|\,f(kgh)=f(g)\text{ for all } k\in K,\ g\in G,\ h\in H\}$$
$$C^\infty_{\text{even}}(A)=\{f\in C^\infty(A)\,|\,f(a_t)=f(a_{-t})\text{ for all } t\in\mathbb{R}\}$$
$$\text{Cl}(A^+)=\{a_t\,|\,t\geqq 0\}.$$

THEOREM 2.6. $G=KAH=K\text{Cl}(A^+)H$. *If* $g\in G$, *then there is a unique* $t\geqq 0$ *such that* $g\in Ka_tH$. *The set* $C^\infty(K\setminus G/H)$ *is in bijective correspondence, via restriction to* A, *with the set* $C^\infty_{\text{even}}(A)$.

PROOF. [Flensted-Jensen, 3], p. 118. ∎

THEOREM 2.7. *Let* $g\in G$. *Then* $P_0(g\cdot x^0)\neq 0$ *if and only if* $g\in NAH\cup wNAH$, *and* $P_0(g\cdot x^0)>0$ *if and only if* $g\in NAH$. *In this latter case, the elements* $n\in N$, $a_t\in A$, $h\in H$ *such that* $g=na_th$, *are unique and* $t=-\frac{1}{2}\log(P_0(g\cdot x^0))$.

Furthermore, the mapping $(n,a,h)\mapsto nah$ of $N\times A\times H$ onto NAH is an analytic diffeomorphism onto this open subset of G. The same is true for the mapping $(n,a,h)\mapsto wnah$ of $N\times A\times H$ onto $wNAH$. The disjoint union $NAH\cup wNAH$ is open and dense in G.

PROOF. [Kosters, 9], p. 94 gives most of this theorem. The fact that the mappings are analytic diffeomorphisms is proved as in [Oshima, Sekiguchi, 13], p. 10. ■

REMARK 2.7.A. If $g=a_t nh$ ($g\in G$, $n\in N$, $t\in\mathbb{R}$, $h\in H$), then $a_{-t}g\cdot x^0=n\cdot x^0$; this guarantees an explicit calculation of n (cf. Chapter 3).

Next we give the Bruhat-decomposition of G. W_1 is a set, consisting of seven elements for $n\geqq 4$; for $n=3$, W_1 has six elements. W_1 will be given explicitly in Chapter 6.

THEOREM 2.8. 1) $G=\bigcup_{\bar{w}\in W_1}\bar{P}\bar{w}\bar{P}$ *(disjoint union).*

2) *If $g\in G$, then $g\in\bar{N}MAN$ if and only if $P(g\cdot\xi^0,\overline{\xi^0})>0$ and $g\in w\bar{N}MAN$ if and only if $P(g\cdot\xi^0,\overline{\xi^0})<0$. If $g=a_t\bar{n}mn$ ($g\in G$, $t\in\mathbb{R}$, $\bar{n}\in\bar{N}$, $m\in M$, $n\in N$), then $t=\frac{1}{2}\log\left(\frac{1}{4}P(g\cdot\xi^0,\overline{\xi^0})\right)$.*

Furthermore, the mapping $(\bar{n},m,a,n)\mapsto\bar{n}man$ is an analytic diffeomorphism of $\bar{N}\times M\times A\times N$ onto $\bar{N}MAN$. The same is true for the mapping $(\bar{n},m,a,n)\mapsto$ $\mapsto w\bar{n}man$ of $\bar{N}\times M\times A\times N$ onto $w\bar{N}MAN$. The disjoint union $\bar{N}MAN\cup$ $\cup\, w\bar{N}MAN$ $(=\bar{N}\bar{M}AN=\bar{N}\bar{P})$ is open and dense in G.

PROOF. 1) For a more precise statement and the proof, see Lemma 6.2.

2) Note that for $\bar{n}\in N$, $t\in\mathbb{R}$, $m\in M$, $n\in N$:

$$P(\bar{n}a_t mn\cdot\xi^0,\overline{\xi^0})=4e^{2t}.$$

In Chapter 3 we shall explicitly compute $\bar{n}$ when $g=\bar{n}ma_t n$ ($g\in G$, $\bar{n}\in\bar{N}$, $m\in M$, $t\in\mathbb{R}$, $n\in N$), so given such a decomposition $\bar{n}$ and a_t are unique. Because $M\cap N=\{e\}$, we derive that $(\bar{n},m,a,n)\mapsto\bar{n}man$ is injective. Using Lemma 2.8 from [Oshima, Sekiguchi, 13] we find that this mapping is also submersive in the neighbourhood of e, from $\bar{N}\times M_e\times A\times N$ to $\bar{N}M_eAN$. Here M_e is the connected component of M containing the identity element e. Because $\bar{N}MAN$ is the disjoint union of two diffeomorphic open sets, like $\bar{N}M_eAN$, the proof is easily completed. ■

THEOREM 2.9. *$G=KMAN$. If $g\in G$, then $g=kman$ for some $k\in K$, $m\in M$, $a\in A$, $n\in N$, where a is unique and k is unique modulo $K\cap M$. Moreover, k depends analytically on g ($k\in K/K\cap M$); the same is true for a.*

PROOF. [Kosters, 9], p. 95; [Varadarajan, 16], p. 293. ■

Due to this last theorem, we can identify $B(G/P;s)$ and $B(K/K\cap M)$, by restriction ($s\in\mathbb{C}$; cf. [Kashiwara et al. 7], and Chapter 6). We also identify $B_1(G/P;s)$ and

$$B_1(K/K\cap M)=\{f\in B(K)|f(km)=f(k) \text{ for all } k\in K,\ m\in K\cap \bar{M}\}$$

and also $B_2(G/P;s)$ and

$$B_2(K/K\cap M)=\{f\in B(K)|f(km)=\chi(m)f(k) \text{ for all } k\in K,\ m\in K\cap \bar{M}\}.$$

Here we defined χ by

$$\chi(m)=\begin{cases} 1 \text{ if } m\in M \\ -1 \text{ if } m\in wM \end{cases}$$

which corresponds to the character χ we introduced before.

Finally we define, for technical reasons, the subgroup

$$\tilde{H}=\left\{\begin{pmatrix} & & 0 \\ & * & \vdots \\ & & 0 \\ 0 & \dots\ 0 & * \end{pmatrix}\in G\right\}=wHw^{-1}$$

of G. Let

$$\tilde{J}=\begin{pmatrix} 1 & & & \theta \\ & \ddots & & \\ & & 1 & \\ \theta & & & -1 \end{pmatrix}=wJw^{-1}.$$

Then $\tilde{H}$ is the set of fixed points of the involution $\tilde{\sigma}$ of G, defined by $\tilde{\sigma}(g)=\tilde{J}g\tilde{J}$ ($g\in G$). Note that $\mathfrak{a}$ is again maximal abelian in $\mathfrak{p}\cap\tilde{\mathfrak{q}}$ (obvious notation). Everything stated for H also holds for $\tilde{H}$, of course with the appropriate modifications.

3. A COMPACTIFICATION OF G/H

In this chapter we shall construct a compact real analytic manifold $\mathbb{X}$ in which G/H is realized as an open set. The construction is similar to those in [Oshima, 10], [Oshima, Sekiguchi, 13] and [Sekiguchi, 15]. The main result is given in Theorem 3.7.

Let $\hat{X}=G\times\mathbb{R}$, provided with the product topology; $\hat{X}$ is a manifold. G acts analytically on $\hat{X}$ by the formula $g\cdot(g',t)=(gg',t)$, for $g,g'\in G$, $t\in\mathbb{R}$. Define $a(t)$ by $a(t)=a_{-\frac{1}{2}\log|t|}$, for $t\in\mathbb{R}$, $t\neq 0$. Consider the following equivalence relation on $\hat{X}$: $(g,t)\sim(g',t')$ if and only if

i) $t>0$, $t'>0$; $ga(t)\equiv g'a(t') \bmod H$ or

ii) $t = t' = 0$; $g \equiv g' \bmod P$ or
iii) $t < 0$, $t' < 0$; $ga(t) \equiv g'a(t') \bmod \tilde{H}$.

Let $\mathbb{X} = \hat{X}/\sim$, and $\pi: \hat{X} \to \mathbb{X}$ the projection. Provide $\mathbb{X}$ with the quotient topology, so π is continuous. G acts on $\mathbb{X}$ by $g(\pi(g', t)) = \pi(gg', t)$ $(g, g' \in G; t \in \mathbb{R})$. For $g \in G$, define

$$U_g^+ = \{\pi(g\bar{n}, t) | \bar{n} \in \bar{N},\ t \in \mathbb{R},\ t > 0\}$$

$$U_g^0 = \{\pi(g\bar{n}, 0) | \bar{n} \in \bar{N}\}$$

$$U_g^- = \{\pi(g\bar{n}, t) | \bar{n} \in \bar{N},\ t \in \mathbb{R},\ t < 0\}$$

$$U_g = U_g^+ \cup U_g^0 \cup U_g^- = \pi(g\bar{N}, \mathbb{R}).$$

Identify $\mathbb{R}^{2n-3}$ and $\bar{N}$ by sending (x, y, z) to $\bar{n}(x, y, z)$, where $x, y \in \mathbb{R}^{n-2}$, $z \in \mathbb{R}$. Define a mapping $\Phi_g: \mathbb{R}^{2n-2} \to U_g$, for fixed $g \in G$, by $\Phi_g(x, y, z, t) = \pi(g\bar{n}(x, y, z), t)$. By definition, Φ_g is surjective.

LEMMA 3.1. *For all $g \in G$, Φ_g is injective.*

PROOF. Suppose that $\pi(g\bar{n}(x, y, z), t) = \pi(g\bar{n}(x', y', z'), t')$. Then there are three cases: i) $t, t' > 0$; ii) $t = t' = 0$; iii) $t, t' < 0$.

i) For some $h \in H$, $\bar{n}(x, y, z)a(t) = \bar{n}(x', y', z')a(t')h$. Theorem 2.7, with N replaced by $\bar{N}$, immediately implies $\bar{n}(x, y, z) = \bar{n}(x', y', z')$ and $a(t) = a(t')$, whence $x = x'$, $y = y'$, $z = z'$ and $t = t'$.

ii) For some $p \in P$, $\bar{n}(x, y, z) = \bar{n}(x', y', z')p$. Using Theorem 2.8 one easily derives $x = x'$, $y = y'$ and $z = z'$.

iii) Analogous to i): replace H by $\tilde{H}$. ■

So Φ_g is bijective for all $g \in G$. In order to show that $\mathbb{X}$ is a manifold, we need the following crucial lemma (cf. [Oshima, Sekiguchi, 13], Lemma 2.8(iii), [Sekiguchi, 15], Lemma 3.3 and also [Kashiwara et al., 7], Lemma 4.1).

LEMMA 3.2. *For all $g_1, g_2 \in G$ the mapping*

$$\Phi_{g_2}^{-1} \cdot \Phi_{g_1} : \Phi_{g_1}^{-1}(U_{g_1} \cap U_{g_2}) \to \Phi_{g_2}^{-1}(U_{g_1} \cap U_{g_2})$$

defines an analytic diffeomorphism between open subsets of $\bar{N} \times \mathbb{R}$ (or $\mathbb{R}^{2n-2}$).

PROOF. Suppose that $(x', y', z', t') = \Phi_{g_2}^{-1} \cdot \Phi_{g_1}(x, y, z, t)$. Write g instead of $g_2^{-1}g_1$. First let $t > 0$. We have:

$$g\bar{n}(x, y, z)a_{-\frac{1}{2}\log t} = \bar{n}(x', y', z')a_{-\frac{1}{2}\log t'}h$$

for some $h \in H$. Applying σ to both sides of this equation we get:

$$\sigma(g)n(-x, -y, z)a_{\frac{1}{2}\log t} = n(-x', -y', z')a_{\frac{1}{2}\log t'}h.$$

Call the left hand side d. In order to compute x', y', z' and t' in terms of x, y, z and t, use Theorem 2.7 and Remark 2.7.A. A careful examination of the matrices $d\cdot x^0$ and $n(-x',-y',z')a_{\frac{1}{2}\log t'}\cdot x^0$ — for example, in order to obtain y', add the first and the last column — leads to:

$$t'=1/(d_{11}-d_{n1})(d_{11}^{-1}+d_{1n}^{-1})$$

$$y_i'=-d_{i1}/(d_{11}-d_{n1}) \qquad (i=2,\dots,n-1)$$

$$x_i'=d_{1i}^{-1}/(d_{11}^{-1}+d_{1n}^{-1}) \qquad (i=2,\dots,n-1)$$

$$-z'+(x',y')=\frac{d_{11}^{-1}}{(d_{11}^{-1}+d_{1n}^{-1})}-\frac{1+t'}{2}=-\frac{d_{1n}^{-1}}{(d_{11}^{-1}+d_{1n}^{-1})}+\frac{1-t'}{2}.$$

Straightforward computation yields, for $i=1,2,\dots,n$:

$$d_{i1}=\sigma(g)_{i1}\left(\frac{\sqrt{t}}{2}+(z+\tfrac{1}{2})\frac{1}{\sqrt{t}}\right)-\frac{1}{\sqrt{t}}\sum_{j=2}^{n-1}\sigma(g)_{ij}y_j+\sigma(g)_{in}\left(\frac{\sqrt{t}}{2}+(z-\tfrac{1}{2})\frac{1}{\sqrt{t}}\right)$$

$$d_{1i}^{-1}=\sigma(g^{-1})_{1i}\left(\frac{\sqrt{t}}{2}+(\tfrac{1}{2}-z+(x,y))\frac{1}{\sqrt{t}}\right)+\frac{1}{\sqrt{t}}\sum_{j=2}^{n-1}\sigma(g^{-1})_{ji}x_j+$$

$$+\sigma(g^{-1})_{ni}\left(-\frac{\sqrt{t}}{2}+(\tfrac{1}{2}+z-(x,y))\frac{1}{\sqrt{t}}\right).$$

From these expressions x',y',z' and t' can be explicitly computed. Note that all denominators involved are nonzero, because of Theorem 2.7.

Now let $t=0$. We have

$$g\bar{n}(x,y,z)=\bar{n}(x',y',z')p$$

for some $p\in P$. Call the left hand side f; let both sides act on ξ^0. Theorem 2.8 implies

$$(f_{11}+f_{1n}+f_{n1}+f_{nn})(f_{11}^{-1}-f_{1n}^{-1}-f_{n1}^{-1}+f_{nn}^{-1})>0,$$

in particular this is nonzero. Furthermore,

$$x_i'=(f_{ni}^{-1}-f_{1i}^{-1})/(f_{11}^{-1}-f_{1n}^{-1}-f_{n1}^{-1}+f_{nn}^{-1}) \qquad (i=2,\dots,n-1)$$

$$y_i'=(f_{in}+f_{i1})/(f_{11}+f_{1n}+f_{n1}+f_{nn}) \qquad (i=2,\dots,n-1)$$

$$\tfrac{1}{2}-z'+(x',y')=(f_{11}^{-1}-f_{n1}^{-1})/(f_{11}^{-1}-f_{1n}^{-1}-f_{n1}^{-1}+f_{nn}^{-1}).$$

For example, x' follows from addition of the first and last row of the matrix $f\cdot\xi^0$. Also, for $i=1,2,\dots,n$:

$$f_{i1}=g_{i1}(z+1)+\sum_{j=2}^{n-1}g_{ij}y_j-g_{in}z$$

$$f_{in}=g_{i1}(z)+\sum_{j=2}^{n-1}g_{ij}y_j+g_{in}(1-z)$$

$$f_{1i}^{-1}=g_{1i}^{-1}(1-z+(x,y))-\sum_{j=2}^{n-1}g_{ji}^{-1}x_j+g_{ni}^{-1}(-z+(x,y))$$

$$f_{ni}^{-1}=g_{1i}^{-1}(z-(x,y))+\sum_{j=2}^{n-1}g_{ji}^{-1}x_j+g_{ni}^{-1}(1+z-(x,y)).$$

Again, x', y' and z' can be explicitly computed in terms of x, y and z.

Finally, let $t<0$. Arguing as before, we get

$$\tilde{\sigma}(g)n(x,y,z)a_{\frac{1}{2}\log|t|}=n(x',y',z')a_{\frac{1}{2}\log|t'|}\tilde{h}$$

for some $\tilde{h}\in\tilde{H}$. Note that $\tilde{\sigma}(\bar{n}(x,y,z))=n(x,y,z)$. Define $\check{x}$, for $x\in\mathbb{R}^{n-2}$, by $(x_1,\ldots,x_{n-2})^{\vee}=(-x_1,x_2,\ldots,x_{n-2})$. Then

$$\sigma(wgw)n(\check{x},-\check{y},-z)a_{\frac{1}{2}\log|t|}=n(\check{x}',-\check{y}',-z)a_{\frac{1}{2}\log|t'|}h$$

where $h\in H$. Tedious calculations give the same formulas as in the case $t>0$, with t replaced by $-t$. In fact, the results for the case $t>0$ can be applied to this last equation.

Note that, for $i,j=1,2,\ldots,n$: $\sigma(g)_{ij}=g_{ij}$ $(i,j\neq 1)$, $\sigma(g)_{i1}=-g_{i1}$, $\sigma(g)_{1j}=-g_{1j}$ and $\sigma(g)_{11}=g_{11}$.

After all these preparations, the lemma can be proved. Take for example y_i'. The formulas derived above easily imply: $(i\in\{2,\ldots,n-1\})$

$$y_i'=\frac{g_{i1}\left(\frac{t}{2}+z+\frac{1}{2}\right)+\sum_{j=2}^{n-1}g_{ij}y_j-g_{in}\left(\frac{t}{2}+z-\frac{1}{2}\right)}{(g_{11}+g_{n1})\left(\frac{t}{2}+z+\frac{1}{2}\right)+\sum_{j=2}^{n-1}(g_{1j}+g_{nj})y_j-(g_{1n}+g_{nn})\left(\frac{t}{2}+z-\frac{1}{2}\right)}$$

for (x,y,z,t) in $\Phi_{g_1}^{-1}(U_{g_1}\cap U_{g_2})$ with $t>0$.

Note however, that this formula is also valid for $t=0$ and $t<0$. This shows, that y_i' is a rational, hence analytic function of x, y, z and t. The crucial point here is the behaviour near $t=0$, of course. Similar computations can be given for x_i' $(i=2,\ldots,n-1)$, z' and t'. Note, that in particular it follows that $\Phi_{g_1}^{-1}(U_{g_1}\cap U_{g_2})$ is open. The details are left to the reader. ■

Here we also completed the proof of Theorem 2.8, as was already mentioned there. If $g\in G$, $g=\bar{n}man$ $(\bar{n}\in\bar{N},\ m\in M,\ a\in A,\ n\in N)$, we write $H_B(g)=\log a$. Then:

COROLLARY 3.3. *Notation as in the proof of Lemma* 3.2.

$$\left.\frac{\partial}{\partial t}\right|_{t=0}t'(g,x,y,z,t)=e^{-2\alpha_0(H_B(g_2^{-1}g_1\bar{n}(x,y,z)))}.$$

PROOF. From the proof of the previous lemma one gets:

$$t'=t/(\sqrt{t}(a_{11}-d_{n1}))(\sqrt{t}(d_{11}^{-1}+d_{1n}^{-1}))$$

where the denominator is analytic. Then it is easy to see that

$$\left.\frac{\partial}{\partial t}\right|_{t=0}t'(g,x,y,z,t)=(\text{denominator in } t=0)^{-1}=$$

$$=4/(f_{11}+f_{1n}+f_{n1}+f_{nn})(f_{11}^{-1}-f_{1n}^{-1}-f_{n1}^{-1}+f_{nn}^{-1})=e^{-2t_0},$$

if $a_{t_0}=\exp(H_B(g_2^{-1}g_1\bar{n}(x,y,z)))$, using Theorem 2.8. ■

COROLLARY 3.4. *Φ_g is a homeomorphism of $\mathbb{R}^{2n-2}$ onto the open subset U_g of $\mathbb{X}$ (for all $g \in G$).*

PROOF. 1) Φ_g is continuous. Let $i: \bar{N} \times \mathbb{R} \to G \times \mathbb{R}$ be defined by $i(\bar{n}, t) = (\bar{n}, t)$ and $\mathfrak{l}_g: G \times \mathbb{R} \to G \times \mathbb{R}$ by $\mathfrak{l}_g(g', t) = (gg', t)$, for $\bar{n} \in \bar{N}$, $t \in \mathbb{R}$, $g, g' \in G$. Now $\Phi_g = \pi \cdot \mathfrak{l}_g \cdot i$ is clearly continuous, being the composition of continuous maps.

2) Φ_g is open. (Cf. [Oshima, Sekiguchi, 13], p. 18).

Let $U \subset \bar{N} \times \mathbb{R}$ be open, $g \in G$. We have to show that $\pi^{-1}(\Phi_g(U))$ is open in $\hat{X}$. Take an arbitrary point (g_0, t_0) in $\pi^{-1}(\Phi_g(U))$. From Lemma 3.2 one gets an open interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ $(\varepsilon > 0)$, such that $\Phi_g^{-1} \Phi_{g_0}(e, V) \subset U$, relatively compact. If Y in $\mathfrak{g}$ is sufficiently near 0, then $\Phi_g^{-1} \Phi_{\exp Y g_0}(e, V) \subset U$. This shows the existence of an open set $\mathfrak{g}_0 \subset \mathfrak{g}$, containing 0, with the property that $\pi(\exp \mathfrak{g}_0 \cdot g_0, V) \subset \Phi_g(U)$. So $\exp \mathfrak{g}_0 \cdot g_0 \times V$ is an open set in $\hat{X}$, containing (g_0, t_0) and contained in $\pi^{-1}(\Phi_g(U))$. ∎

LEMMA 3.5. *$\mathbb{X}$ is connected and compact.*

PROOF. It is obvious that $\mathbb{X}$ is connected, because $\hat{X}$ has that property. In order to show that $\mathbb{X}$ is compact, it is sufficient to prove that $\mathbb{X} = \pi(K \times [-1, 1])$. Let $g \in G$, $t \in \mathbb{R}$. If $t = 0$ then there exists a $k \in K$ such that $g \in kMAN$, because of Theorem 2.9; so $\pi(g, 0) = \pi(k, 0)$. If $t > 0$, Theorem 2.6 implies the existence of $k \in K$ and $s \geqq 0$ with $ga(t) \in ka_sH$; so $\pi(g, t) = \pi(k, e^{-2s})$. Note that $e^{-2s} \leqq 1$. If $t < 0$, a similar argument applies. So the lemma is proved. ∎

As a consequence, $\mathbb{X}$ is second countable.

LEMMA 3.6. *$\mathbb{X}$ is a Hausdorffspace.*

PROOF. Take x, x' in $\mathbb{X}$. Note that for fixed $g \in G$, U_g and U_{gw} are two disjoint open sets in $\mathbb{X}$, both homeomorphic to the Hausdorffspace $\bar{N} \times \mathbb{R}$. That the sets are disjoint is proved in the following way: $g\bar{n}aH = gw\bar{n}'a'H$ $(\bar{n}, \bar{n}' \in \bar{N}, a, a' \in A)$ contradicts Theorem 2.7 (with N replaced by $\bar{N}$), and Theorem 2.8 deals with the equation $gMAN = gwMAN$. Define $G_x = \{g \in G \mid x \in U_g \cup U_{gw}\} \subset G$ and the same notion for x'. It is easy to see that G_x is open and dense in G. Indeed, suppose that $t > 0$. For appropriate $g' \in G$, $x = \pi(g', 1)$. Then $x \in U_g \cup U_{gw}$ if and only if $g^{-1} \in (\bar{N}AH \cup w\bar{N}AH)g'^{-1}$, which is open and dense in G, according to Theorem 2.7. If $t = 0$, $x = \pi(g', 0)$ for some $g' \in G$. Now $x \in U_g \cup U_{gw}$ if and only if $g^{-1} \in (\bar{N}MAN \cup w\bar{N}MAN)g'^{-1}$, which is open and dense in G by Theorem 2.8. Hence $G_x \cap G_{x'} \neq \emptyset$. Take $g \in G_x \cap G_{x'}$, then $x, x' \in U_g \cup U_{gw}$. Now x and x' are easily separated by open sets. ∎

THEOREM 3.7. *$\mathbb{X}$ is a compact connected real analytic manifold. G acts real analytically on $\mathbb{X}$. There are three G-orbits in $\mathbb{X}$:*

$$\mathbb{X}^+ = \bigcup_{g \in G} U_g^+, \quad \mathbb{X}^0 = \bigcup_{g \in G} U_g^0, \quad \mathbb{X}^- = \bigcup_{g \in G} U_g^-.$$

$\mathbb{X}^+$ is diffeomorphic to G/H, $\mathbb{X}^0$ to G/P and $\mathbb{X}^-$ to $G/\tilde{H}$. The mappings Φ_g $(g \in G)$ give an atlas on $\mathbb{X}$.

PROOF. We only have to remark that for $g \in G$,

$$\tau_g : (x, y, z, t) \mapsto g\bar{n}(x, y, z)a(t)H \quad ((x, y, z, t) \in \mathbb{R}^{2n-2}, t > 0)$$

establish real analytic diffeomorphisms from $\bar{N} \times \mathbb{R}$ onto open subsets of G/H. The other statements have already been proved in this chapter. ∎

REMARK. Using this theorem, we often identify $\mathbb{X}^+$ and G/H.

4. THE BOUNDARY VALUE MAP

In this chapter we construct a differential operator $\Box$ on the manifold $\mathbb{X}$. This operator comes from the Casimir-element Ω, which is in the centre of $U(\mathfrak{g})$. Using the special form of $\Box$ in local coordinates on $\mathbb{X}$, we are able to apply the theory of Kashiwara and Oshima on regular singularities in order to define the boundary value map β_s. The main result of this chapter is Theorem 4.3.

First we construct Ω. For $i \in \{2, \ldots, n-1\}$, let $H_i = E_{11} - 2E_{ii} + E_{nn}$, $H_0 = E_{n1} + E_{1n}$, $X_i = E_{i1} - E_{in}$, $Y_i = E_{1i} + E_{ni}$, $U_i = E_{i1} + E_{in}$, $V_i = E_{1i} - E_{ni}$, $X_0 = E_{11} + E_{n1} - E_{1n} - E_{nn}$ and $Y_0 = E_{11} + E_{1n} - E_{n1} - E_{nn}$. It is easily seen that $\{E_{ij}, H_i | i, j = 2, \ldots, n-1; i \neq j\}$ is a basis for $\mathfrak{m}$, $\{H_0\}$ for $\mathfrak{a}$, $\{X_i, Y_i | i = 2, \ldots, n-1\}$ for $\mathfrak{g}(\alpha_0)$, $\{U_i, V_i | i = 2, \ldots, n-1\}$ for $\mathfrak{g}(-\alpha_0)$, $\{X_0\}$ for $\mathfrak{g}(2\alpha_0)$ and $\{Y_0\}$ for $\mathfrak{g}(-2\alpha_0)$. An easy computation shows that modulo $U(\mathfrak{g})\mathfrak{h}$:

$$\Omega \equiv \frac{1}{2n} \{\tfrac{1}{2}H_0^2 + \tfrac{1}{4}(X_0 Y_0 + Y_0 X_0) + \tfrac{1}{2} \sum_{i=2}^{n-1} (X_i V_i + V_i X_i + Y_i U_i + U_i Y_i)\}.$$

For a definition of Ω, consult [Warner, 18], p. 168. Simple computations yield:

$$4n\Omega \equiv H_0^2 + 2(n-1)H_0 - Y_0^2 - 2(n-2)Y_0 + 4 \sum_{i=2}^{n-1} V_i U_i \text{ (modulo } U(\mathfrak{g})\mathfrak{h}).$$

As usual, we consider elements of $U(\mathfrak{g})$ as left invariant differential operators on G: for $Y \in \mathfrak{g}$, $f \in B(G)$, $g \in G$ we let

$$(Yf)(g) = \frac{d}{dt}\bigg|_{t=0} f(g \exp tX),$$

which has a natural extension to $U(\mathfrak{g})$. From now on, identify $B(G/H)$ and the set of right H-invariant hyperfunctions on G. For $u \in B(G)$, write

$$u^g(x, y, z, t) = u(g\bar{n}(x, y, z)a(t)) \qquad (a(0) = e)$$

where $g \in G$; $x, y, z, t \in \mathbb{R}^{2n-2}$. We have:

LEMMA 4.1. *Let $u \in B(G/H)$ and $t > 0$. Then:*

$$(4n\Omega u)^g(x, y, z, t) = \left\{4\left(t\frac{\partial}{\partial t}\right)^2 - 4(n-1)t\frac{\partial}{\partial t} - t^2\frac{\partial^2}{\partial z^2} + \right.$$

$$+4t\sum_{i=2}^{n-1}\frac{\partial^2}{\partial x_i\partial y_i}+2(n-2)t\frac{\partial}{\partial z}+4t\frac{\partial}{\partial z}\sum_{i=2}^{n-1}x_i\frac{\partial}{\partial x_i}\Big\}\,u^g(x,y,z,t).$$

PROOF.

1) $$(H_0u)(g\bar{n}(x,y,z)a(t))=\frac{d}{ds}\Big|_{s=0}u(g\bar{n}(x,y,z)a(te^{-2s}))=-2t\frac{\partial}{\partial t}u^g(x,y,z,t).$$

2) $$(Y_0u)(g\bar{n}(x,y,z)a(t))=\frac{d}{ds}\Big|_{s=0}u(g\bar{n}(x,y,z)\bar{n}(0,0,s))=$$

$$=\frac{d}{ds}\Big|_{s=0}u(g\bar{n}(x,y,z+st)a(t))=t\frac{\partial}{\partial z}u^g(x,y,z,t).$$

3) $$(V_iu)(g\bar{n}(x,y,z)a(t))=\frac{d}{ds}\Big|_{s=0}u(g\bar{n}(x,y,z)\bar{n}(s\sqrt{t}\,e_i,0,0)a(t))=$$

$$=\sqrt{t}\frac{\partial}{\partial x_i}u^g(x,y,z,t),\ \text{where } e_i=(0,\ldots,0,1,0,\ldots,0),\ i^{\text{th}}\text{ entry is }1.$$

4) $$(U_iu)(g\bar{n}(x,y,z)a(t))=\frac{d}{ds}\Big|_{s=0}u(g\bar{n}(x,y,z)\bar{n}(0,s\sqrt{t}\,e_i,0)a(t))=$$

$$=\frac{d}{ds}\Big|_{s=0}u(g\bar{n}(x,y+s\sqrt{t}\,e_i,z+s\sqrt{t}\,x_i)a(t))=$$

$$=\left(\sqrt{t}\,x_i\frac{\partial}{\partial z}+\sqrt{t}\frac{\partial}{\partial y_i}\right)u^g(x,y,z,t).$$

Substitution of 1), 2), 3) and 4) in the expression already obtained for $4n\Omega$, completes the proof. ∎

Note that a similar computation for $G/\tilde{H}$ gives the same result, with $t<0$. In fact

$$4n\Omega\equiv H_0^2+2(n-1)H_0-Y_0^2+2(n-2)Y_0-4\sum_{i=2}^{n-1}V_iU_i\ (\text{modulo } U(\mathfrak{g})\tilde{\mathfrak{h}})$$

and

$$H_0 \text{ corresponds to } -2t\frac{\partial}{\partial t},\ Y_0 \text{ to } -t\frac{\partial}{\partial z},\ V_i \text{ to } \sqrt{|t|}\frac{\partial}{\partial x_i}$$

and

$$U_i \text{ to } \sqrt{|t|}\left(x_i\frac{\partial}{\partial z}+\frac{\partial}{\partial y_i}\right).$$

Now we can define a global differential operator $\square$ on $\mathbb{X}$. Use the charts (Φ_g, U_g) as defined in Chapter 3. On U_g^+, $\square$ is given by the expression in Lemma 4.1. However, this expression extends to U_g. Notice that the coefficients are analytic in x, y, z and t. In this way, $\square$ becomes a differential operator on $\mathbb{X}$, such that its restriction to $\mathbb{X}^+$, again denoted by $\Box$, coincides with $4n\Omega$. It is easy to see that $\Box$ commutes with the G action on $\mathbb{X}$. Indeed, it is sufficient to check this on the U_g ($g \in G$), and for $t \neq 0$; there it follows from the invariance of Ω.

LEMMA 4.2. *Every G-invariant differential operator on $\mathbb{X}$ with real analytic coefficients is a polynomial in $\Box$.*

PROOF. As remarked in Chapter 2, every G-invariant differential operator on G/H is a polynomial in $\Box|_{\mathbb{X}^+}$. Notice, that a differential operator on $\mathbb{X}$ with real analytic coefficients is uniquely determined by its restriction to $\mathbb{X}^+$. This follows from the fact that a real analytic function on $\mathbb{R}^n$, which is zero on

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n | x_n > 0\}, \text{ is zero everywhere.} \qquad \blacksquare$$

Consider the following differential equation on $\mathbb{X}$ ($s \in \mathbb{C}$, $u \in B(\mathbb{X})$)

$$\Box u = (s^2 - \varrho^2)u \qquad (M_s)$$

and let

$$B(G/H; M_s) = \{f \in B(\mathbb{X}^+) | f \text{ satisfies } (M_s)\}.$$

Now we use the theory of Kashiwara and Oshima. As a reference we take [Oshima, Sekiguchi, 13], 2.2. where the results of [Kashiwara, Oshima, 8] are stated in the way we need them. First note that the differential equation (M_s), containing the holomorphic parameter s, has regular singularities in the weak sense along $\mathbb{X}^0$. This is a direct consequence of Lemma 4.1. The characteristic exponents are the complex solutions of the equation $4\lambda^2 - 4\varrho\lambda - s^2 + \varrho^2 = 0$, so they are $\frac{1}{2}(\varrho + s)$ and $\frac{1}{2}(\varrho - s)$. The difference of these numbers equals s. In order to define the boundary value map, we have to make the change of variables $t = \tilde{t}^2$. Then (M_s) becomes a differential equation with regular singularities in the strong sense, which means among other things that

$$\tilde{t}^2 \frac{\partial^2}{\partial x_2 \partial y_2}$$

occurs instead of

$$t \frac{\partial^2}{\partial x_2 \partial y_2}.$$

However, this is not a real analytic change of coordinates near $t = 0$. Nevertheless, the theory can still be applied. The characteristic exponents also change; they become $\varrho + s$ and $\varrho - s$, so their difference turns out to be $2s$. From now on, assume $2s \notin \mathbb{Z}$.

Take $u \in B(G/H; M_s)$ and $g \in G$. Consider $u|_{U_g^+}$. Let $\tilde{u}$ be an extension of $u|_{U_g^+}$ to U_g, with supp $(\tilde{u}) \subset U_g^+ \cup U_g^0$ and such that $\tilde{u}$ satisfies (M_s) on U_g. Let $sp(\tilde{u})$ be the microfunction corresponding to $\tilde{u}$. Then there exist certain microdifferential operators R_i $(i=1,2)$ and hyperfunctions $\phi_i(s,x,y,z)$ on U_g^0 $(i=1,2)$ such that:

$$sp(\tilde{u}) = R_1\ sp(\phi_1(s,x,y,z)\tilde{t}_+^{\varrho-s}) + R_2\ sp(\phi_2(s,x,y,z)\tilde{t}_+^{\varrho+s}).$$

For these facts, the reader is referred to [Oshima, Sekiguchi, 13], 2.2. The ϕ_i $(i=1,2)$ are called the boundary values of $u|_{U_g^+}$. We denote them by:

$$\phi_1(s,\cdot) = \beta_s^g u(\cdot);\ \phi_2(s,\cdot) = \beta_{-s}^g u(\cdot).$$

Note, that the systems (M_s) and (M_{-s}) coincide. Theorem 2.14 from [Oshima, Sekiguchi, 13] implies that this definition is independent of the choice of g, in fact (with $g' \in G$):

$$(\beta_s^g u)(x,y,z)(dt)^{(\varrho-s)/2} = (\beta_s^{g'} u)(x',y',z')(dt')^{(\varrho-s)/2}$$

defines a hyperfunction valued section in the linebundle $T^*_{\mathbb{X}^0}\mathbb{X}$, where the equality holds for $\pi(g\bar{n}(x,y,z),t) = \pi(g'\bar{n}(x',y',z'),t')$. If $g_1 \in G$, $g_1 \in \bar{N}MAN$, define $\bar{n}_B(g_1)$ and $H_B(g_1)$ by

$$g \in \bar{n}_B(g_1)M \exp H_B(g_1)N,\ \bar{n}_B(g_1) \in \bar{N},\ H_B(g_1) \in \mathfrak{a}.$$

Using Corollary 3.3, that states

$$dt' = e^{-2\alpha_0(H_B(g'^{-1}g\bar{n}))}dt$$

we derive, that for $\bar{n} \in \bar{N} \cap g^{-1}g'\bar{N}P$:

$$(\beta_s^g u)(\bar{n}) = (\beta_s^{g'} u)(\bar{n}_B(g'^{-1}g\bar{n}))\ \exp\ ((s-\varrho)\alpha_0(H_B(g'^{-1}g\bar{n}))). \qquad (*)$$

Now we can define a hyperfunction on G:

$$(\beta_s u)(g_0) = (\beta_s^g u)(\bar{n}_B(g^{-1}g_0))\ \exp\ ((s-\varrho)\alpha_0(H_B(g^{-1}g_0)))$$

for $g_0 \in g\bar{N}P$, which is open in G. We have to show, that this definition is independent of the choice of g. Therefore, suppose that $g_0 \in g\bar{N}P \cap g'\bar{N}P$ for $g, g' \in G$. An application of (*) yields:

$$\begin{aligned}(\beta_s u)(g_0) &= (\beta_s^{g'} u)(\bar{n}_B(g'^{-1}g\bar{n}_B(g^{-1}g_0)))\ \exp\ ((s-\varrho)\alpha_0(H_B(g'^{-1}g\bar{n}_B(g^{-1}g_0))) \\ &= (\beta_s^{g'} u)(\bar{n}_B(g'^{-1}g_0))\ \exp\ ((s-\varrho)\alpha_0(H_B(g'^{-1}g_0))\end{aligned}$$

proving our claim. (Note that $\bar{n}_B(g^{-1}g_0) \in g^{-1}g'\bar{N}P$.) It is easy to see, that $\beta_s u \in B(G/P;s)$, because

$$(\beta_s u)(g_0 m a_t n) = (\beta_s^g u)(\bar{n}_B(g^{-1}g_0))\ \exp\ ((s-\varrho)\alpha_0(H_B(g^{-1}g_0) + H_B(a_t)))$$

for $m \in M$, $t \in \mathbb{R}$, $n \in N$, $g, g_0 \in G$, $g_0 \in g\bar{N}P$.

So β_s is a linear map from $B(G/H;M_s)$ into $B(G/P;s)$. It is called the boundary value map.

Notice that both spaces are invariant under the action of G:

$$(\pi_s(g)f)(x) = f(g^{-1}x) \quad (g \in G, x \in G/H; f \in B(G/H; M_s))$$

$$(\tilde{\pi}_s(g)f)(x) = f(g^{-1}x) \quad (g \in G, x \in G/H; f \in B(G/P; s)).$$

THEOREM 4.3. *Let $2s \notin \mathbb{Z}$. Then β_s is a linear, G-equivariant mapping from $B(G/H; M_s)$ into $B(G/P; s)$.*

PROOF. Let $u \in B(G/H; M_s)$. We have to show, that for fixed $g \in G$:

$$(\tilde{\pi}_s(\exp\ Y)\beta_s u)(g \exp X) = \beta_s(\pi_s(\exp\ Y)u)(g \exp X) \qquad (\#)$$

for all X, Y in $\mathfrak{g}$ sufficiently near 0. The left hand side of (#), for X, Y such that $\exp - Y\ g \exp X \in g\bar{N}P$, is equal to

$$(\beta_s^g u)(\bar{n}_B(g^{-1} \exp - Y\ g \exp X)) \exp((s-\varrho)\alpha_0(H_B(g^{-1} \exp - Y\ g \exp X))).$$

It is easy to see, that

$$\beta_s^g(\pi_s(\exp\ Y)u)(\bar{n}) = (\beta_s^{\exp - Yg}u)(\bar{n}),$$

because for both sides we have to consider the function

$$(x, y, z, t) \mapsto u(\exp - Yg\bar{n}(s, x, y)a(t)H).$$

So the right hand side of (#) is equal to:

$$(\beta_s^{\exp - Yg}u)(\bar{n}_B(g^{-1}g \exp X)) \exp((s-\varrho)\alpha_0(H_B(g^{-1}g \exp X))).$$

Now the two expressions obtained are equal, because we can use formula (*), which is valid for X, Y sufficiently near 0, and some simple properties of $\bar{n}_B$ and H_B. ■

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