Combinatorial Game Theory
From Conway to Nash

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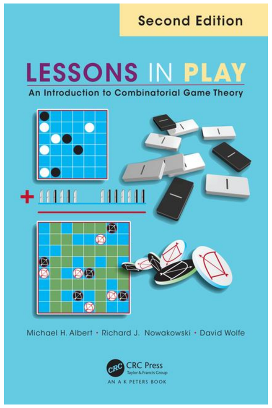
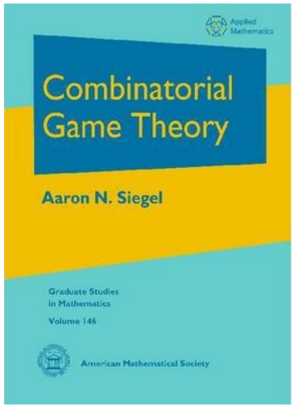
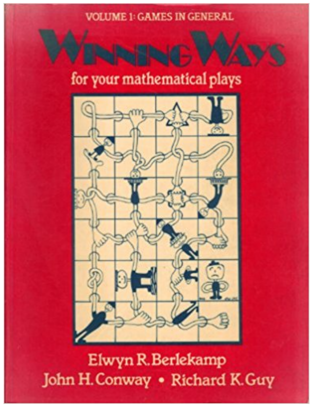
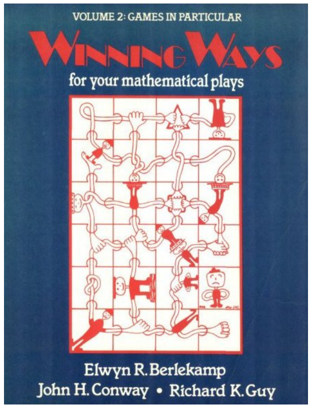
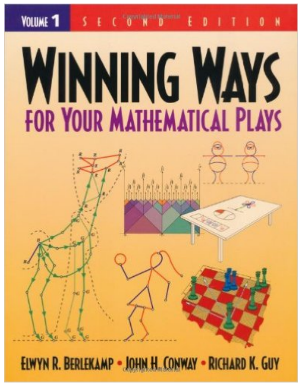
Combinatorial Game Theory:

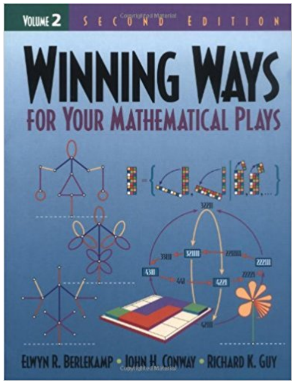


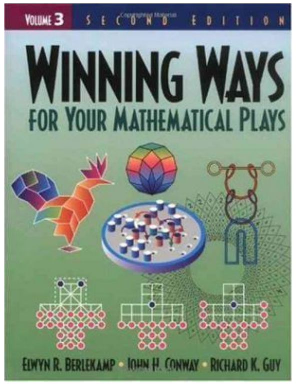


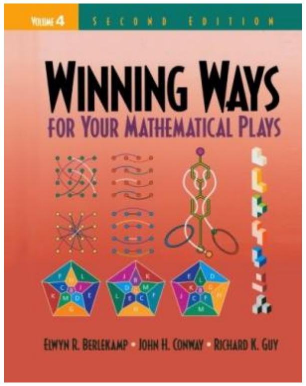












Three main references:

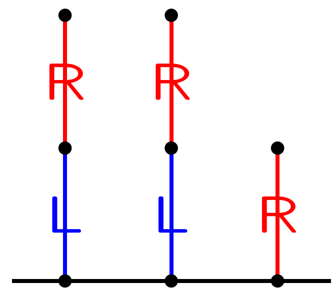
LessonsInPlay:
M.H. Albert, R.J. Nowakowski and D. Wolfe, Lessons in Play, **second edition**, CRC Press, 2019.

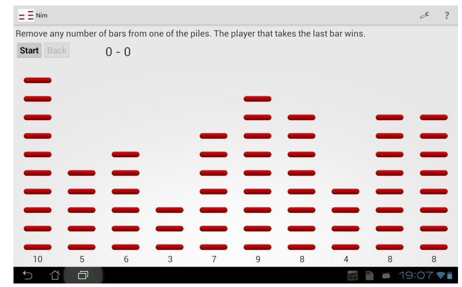
Siegel:
A.N. Siegel, Combinatorial Game Theory, AMS, 2013.

WinningWays:
E.R. Berlekamp, J.H. Conway and R.K. Guy, Winning Ways for your Mathematical Plays, 1982/2001.
(Note that there are two editions: the first has two volumes, the second has four volumes.)

First we examine two combinatorial games (two players, perfect information, no chance):

- Hackenbush
- Nim

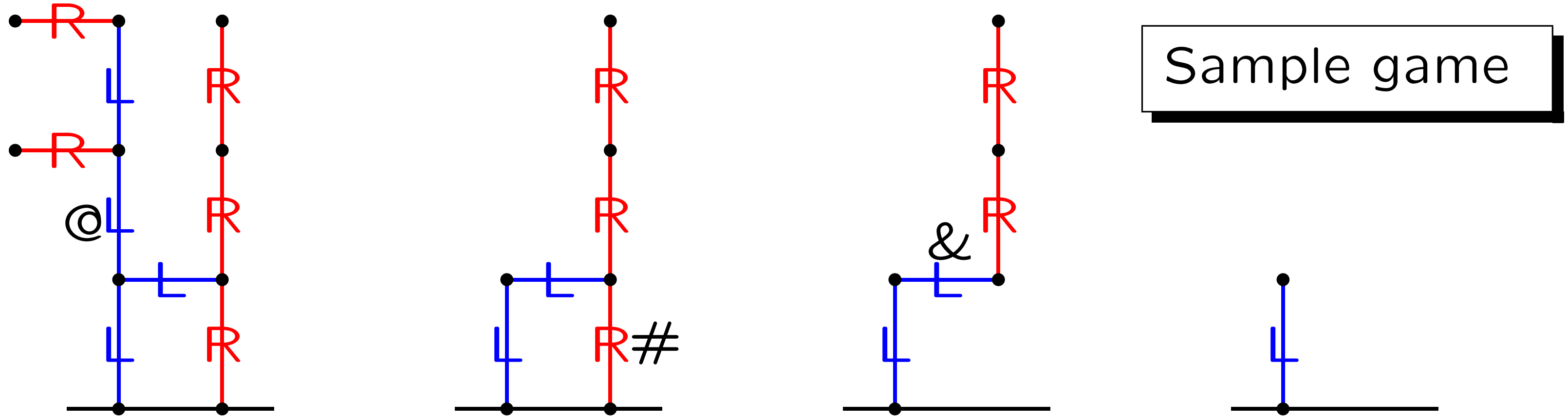




And then briefly:

- Synchronized versions
- How about Nash equilibria?

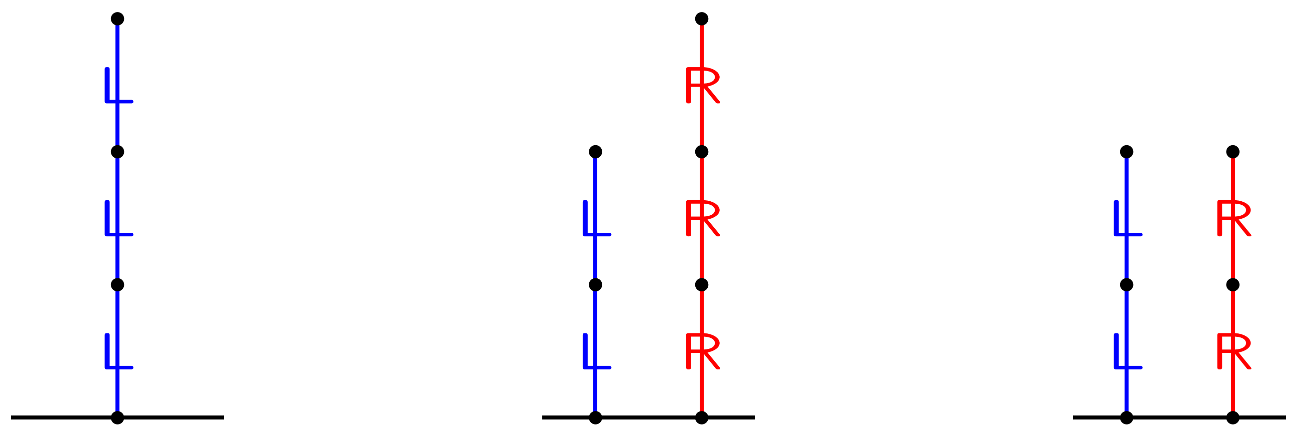
In the game (Blue-Red-)Hackenbush the two players Left = she and Right = he alternately remove a bLue or a Red edge. All edges that are no longer connected to the ground, are also removed. *If you cannot move, you lose!*



Left chooses @, Right # (stupid), Left &. Now Left wins because Right cannot move.

By the way, Right can win here, whoever starts!

When playing Hackenbush, what is the value of a position?



value 3 value $2 - 3 = -1$ value $2 - 2 = 0$

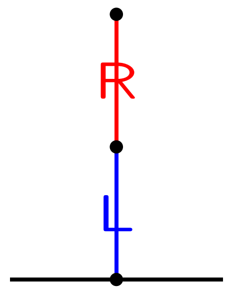
value > 0: Left wins (whoever starts) $\mathcal{L}$

value $= 0$: first player loses $\mathcal{P}$

value < 0: Right wins (whoever starts) $\mathcal{R}$

Remarkable: Hackenbush has no "first player wins"! $\mathcal{N}$

But what is the value of this position?



If Left begins, she wins immediately. If Right begins, Left can still move, and also wins. So Left always wins. Therefore, the value is > 0.

Is the value equal to 1?

If the value x in the left hand side position would be 1, the value of the right hand side position would be $1+(-1)=0$, and the first player should lose. Is this true?

No! If Left begins, Left loses, and if Right begins Right can also win. So Right always wins (i.e., can always win), and therefore the right hand side position is < 0, and $x+(-1)<0$, and the left one is between 0 and 1.

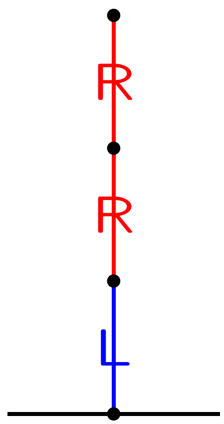
The left hand side position is denoted by $\{\, 0 \mid 1 \,\}$.

Note that the right hand side position does have value 0: the first player loses. And so we have:

$$\{\, 0 \mid 1 \,\} + \{\, 0 \mid 1 \,\} + (-1) = 0,$$

and "apparently" $\{\, 0 \mid 1 \,\} = 1/2$.

We denote the value of a position where Left can play to (values of) positions from the set L and Right can play to (values of) positions from the set R by $\{\, L \mid R \,\}$.

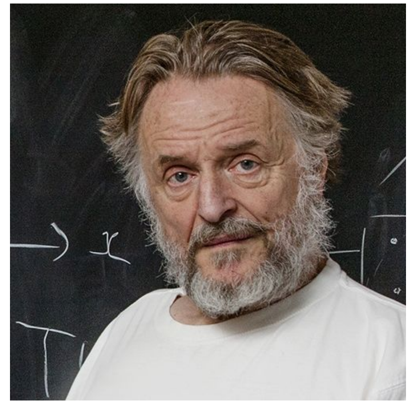


The value is $\{\, 0 \mid \frac{1}{2}, 1 \,\} = \frac{1}{4}$.

Simplicity rule:The value is always the "simplest" number between left and right set: the smallest integer — or the dyadic number with the lowest denominator (power of 2).

Give a position with value 3/8.

Show that $\{\, 0 \mid 100 \,\} = 1$.



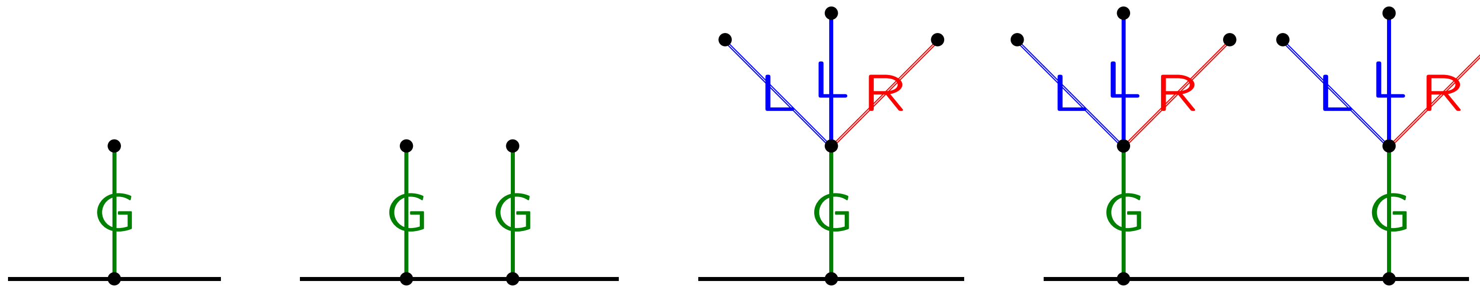
Donald E.(Ervin) Knuth
1938, US
NP; KMP
$\TeX$
change-ringing; 3:16
The Art of Computer Programming

John H.(Horton) Conway
1937–2020, UK $\rightarrow$ US
Co_1, Co_2, Co_3
Doomsday algoritme
game of Life; Angel problem
Winning Ways for your Mathematical Plays

Surreal numbers

$$\varepsilon \cdot \omega = \{\, 0 \mid 1/2, 1/4, 1/8, \ldots \,\} \cdot \{\, 0, 1, 2, 3, \ldots \mid \,\} = 1$$

In Red-Green-Blue-Hackenbush we also have Green edges, that can be removed by both players.



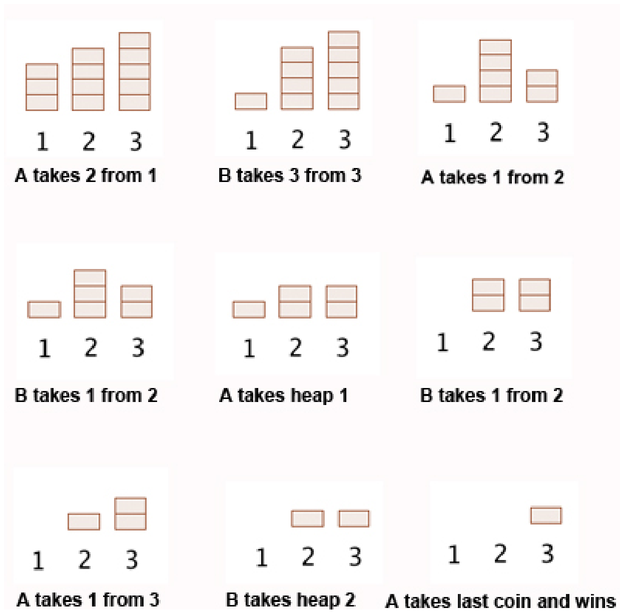
The first position has value $* = \{ 0 \mid 0 \}$ (no surreal number), because the player to move can win: it is in $\mathcal{N}$.

The second position is $* + * = 0$ (player to move loses).

The third position is a first player win.

The fourth position is a win for Left (whoever begins), and is therefore > 0.

In the Nim game we have several stacks of tokens = coins = matches. A move consists of taking a nonzero number of tokens from one of the stacks. If you cannot move, you lose ("normal play").



The game is impartial: both players have the same moves. (And for the misère version: if you cannot move, you win.)

For Nim we have Bouton's analysis from 1901.

We define the nim-sum $x \oplus y$ of two positive integers x en y as the bitwise XOR of their binary representions: addition without carry. With two stacks of equal size the first player loses ($x \oplus x = 0$): use the "mirror strategy".

A nim game with stacks of sizes $a_1, a_2, \ldots, a_k$ is a first player loss exactly if $a_1 \oplus a_2 \oplus \ldots \oplus a_k = 0$. And this sum is the Sprague-Grundy value.

We denote a nim game with value m by $*m$ (the same as a stalk of m green Hackenbush edges). And $*1 = *$. So if $m \neq 0$ the first player loses.

The Sprague-Grundy Theorem roughly says that every impartial game is a Nim game.

With stacks of sizes 29, 21 and 11, we get $29 \oplus 21 \oplus 11 = 3$:

```
11101     29
10101     21
 1011     11
-----     --
00011      3
```

So a first player win, with unique winning move $11 \to 8$.

Why this move, and why is it unique?

How to add these “games” (we already did)? Like this:

$$a + b = \{\, A_L + b, a + B_L \mid A_R + b, a + B_R \,\}$$

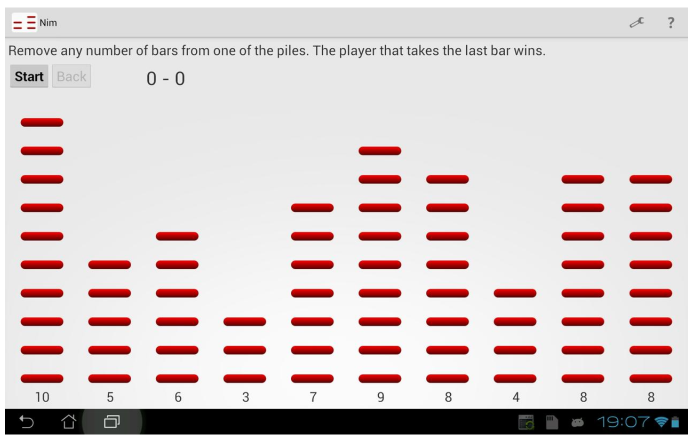
if $a = \{\, A_L \mid A_R \,\}$ and $b = \{\, B_L \mid B_R \,\}$.
We put $u + \emptyset = \emptyset$ and (more general) $u + V = \{u + v : v \in V\}$.

This corresponds with the following: you play two games in parallel, and in every move you must play in exactly one of these two games: the disjunctive sum.

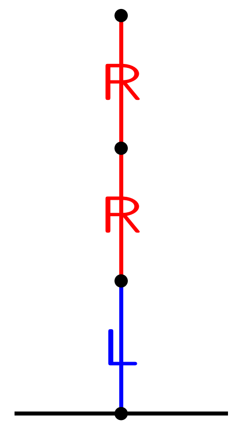
> Verify that $1 + \frac{1}{2} = \{\, 1 \mid 2 \,\} = \frac{3}{2}$.

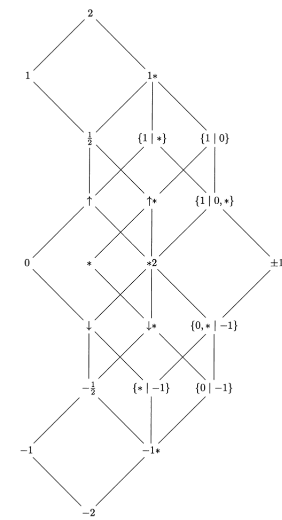
See Claus Tøndering’s paper.

Now consider this addition of two game positions, with on the left a Nim position and on the right a Hackenbush position:



$+$





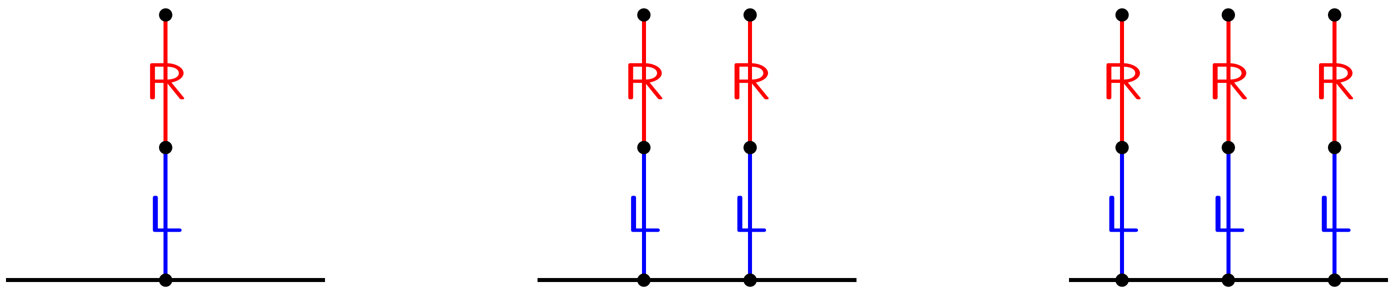
$\{\, 0 \mid * \,\} = \uparrow$

from Siegel

Then this sum is > 0, it is a win for Left!

And we even have: $*m + 1/2^{10000} > 0$, and therefore $-1/2^{10000} < * < 1/2^{10000}$, but $*$ is not comparable to 0.

In Synchronized Hackenbush the players simultaneously remove an edge. If a player cannot move, the other wins with the number of remaining edges as outcome.



The first position H has value = outcome 0.

If the two players play randomly, the second position $2 \cdot H$ has value $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1 = 1/2$.

If the two players play randomly at first, and then clever, the third position $3 \cdot H$ has value $\frac{1}{3} \cdot 1/2 + \frac{2}{3} \cdot 1 = 5/6$.

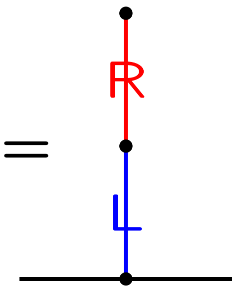
The strategy of a player is a probability distribution over their own moves..

We have a Nash equilibrium if for both players it holds that they cannot improve their result by “unilateral deviation” (i.e., if you deviate from your strategy while the other player sticks to their own, it does not get better for you).

The corresponding value = outcome is “the” Nash value $\nu(G)$ of the game G.

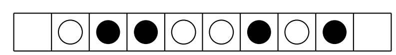
Problem: how to deal with green edges in Hackenbush, or with simultaneous moves on the same Nim stack?

Put $v_n = \nu(n \cdot H)$ and $d_n = v_n - v_{n-1}$.
Then $v_1 = 0$, $v_2 = 1/2$ and $d_2 = 1/2$.

$H =$ 

Theorem (Mark van den Bergh):
We have $v_n = \left((n-1)(1+v_{n-2}) + v_{n-1}\right)/n$ for $n \geq 3$.

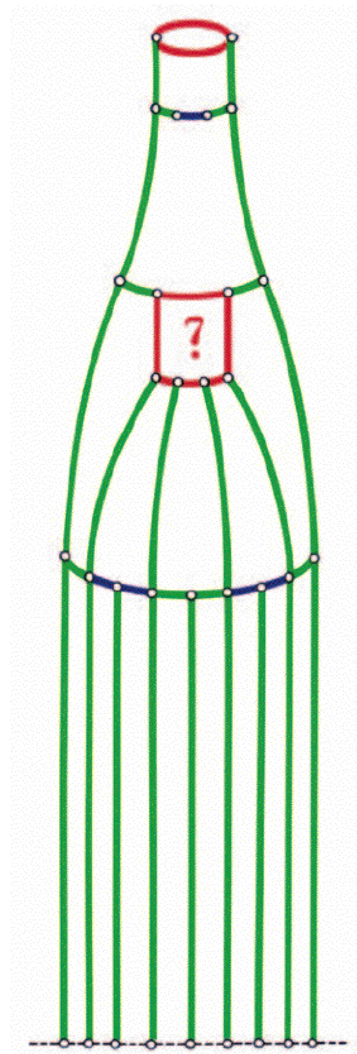
And for $n \geq 3$: $d_n = \frac{1}{2} + \frac{(-1)^n - 1}{4n} \to 1/2$ if $n \to \infty$.

And many partial results on $\nu(m \cdot H - n \cdot H)$, “flowers”, other Hackenbush variants, . . . , Cherries: 

Conjecture: $\nu(n \cdot G) - \nu((n-1) \cdot G) \to \mathrm{CG}(G)$ if $n \to \infty$, for any G, with $\mathrm{CG}(G) = G$'s combinatorial game value.

Combinatorial
Game Theory

Synchronized
games



deep results

Conway → Nash

read LessonsInPlay

← from WinningWays

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