$(f_{4(4)} , so(4,5))$

A survey of results and problems

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1. NOTATIONS AND GENERAL REMARKS

Definitions and notations are as in :

G.van Dijk, Nilpotent orbits on the tangent space of a rank one symmetric space, Report no.13, University of Leiden, 1983.

This paper is referred to as (D). In particular we shall use (D), pp.4-17. Notice that (D), Chapter 5 is not quite correct. In fact, the orbit $H \,.\, \tilde{A}^{e_0+e_4}$ does admit a non-zero H-invariant measure (cf. (D), p.21). We shall return to this later on. Also, formula (2) on p.5 of (D) needs some minor corrections.

So in this paper G is a real connected simple Lie group, whose Lie algebra is of type $f_{4(4)}$. H is a closed connected Lie subgroup of G, which is a two-fold cover of $SO_o(4,5)$.

Let X = G / H. We are interested in the following items :

(1) Give a (nice) model for X.

(2) Determine $D'_{\lambda,H}(X)$ for complex λ. Here $D'_{\lambda,H}(X)$ consists of all H-invariant distributions on X that satisfy $\square\, T = \lambda T$, where $\square$ is the Laplace-Beltrami operator on X. Elements of this space are called spherical distributions.

(3) Determine a Plancherel formula for X. This means a spectral decomposition of the self-adjoint extension $\tilde{\square}$ of $\square$, from $L^2(X)$ to $L^2(X)$. Here $L^2(X)$ denotes the space of functions on X that are square integrable with respect to a non-zero G-invariant measure dx on X. This measure is unique up to a scalar.

(4) Interpretation of (3). Determine the (ir)reducibility of the discrete series, construct representations, compute c-functions and so on.

In this paper we shall deal with (1), (2) and (3). Proofs are not given here. In the last chapter we will comment on (4) and mention some of the problems that are to be expected.

2. THE MODEL

Let X' denote the manifold $\{ Y \in J_{2,1} \mid \text{trace } Y = 1,\ Y^2 = Y \}$. Here $J_{2,1}$ is the real 27-dimensional vectorspace of hermitean 3×3 matrices with coefficients in O^*, the octonions (cf. (D), p.5). G clearly acts on X'. Indeed, one uses the fact that G is contained in the group of automorphisms of $J_{2,1}$. The action is even transitive and the stabilizer of $x^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in G equals H. Therefore X and X' are diffeomorphic. From now on we shall identify them.

Next we examine the H-orbit structure on X. For that purpose we let $Q(x) = \text{trace}\,(\, x\, x^0\,) = x_{11}$ $(x \in X)$. Notice that Q is an H-invariant real analytic function on X, with $Q(X) = \mathbb{R}$ and $Q(x^0) = 1$. Write for real c :

$X(c) = \left\{ x \in X \mid Q(x) = c \right\}$. We have :

LEMMA 1 (i) If $c \neq 0,1$ then $X(c)$ is a 15-dimensional H-orbit.

(ii) $X(0)$ consists of two H-orbits of dimension 8 and 15 resp..

(iii) $X(1)$ consists of three H-orbits, namely $\left\{ x^0 \right\}$, A and B, of dimension 0, 11 and 15 resp..

Because the H-orbit A plays a central role in the following, let us give its definition :

$$A = \left\{ \begin{pmatrix} 1 & y & \bar{x} \\ \bar{y} & 0 & 0 \\ x & 0 & 0 \end{pmatrix} \mid x, y \in O^*,\ (x|x) = (y|y) = xy = 0 \right\} .$$

3. THE SPHERICAL DISTRIBUTIONS

Let dx be a fixed non-zero G-invariant measure on X. If $f \in D(X)$ we define the function Mf on $\mathbb{R}$ by

$$\int_X f(x)\, F(Q(x))\, dx = \int_{\mathbb{R}} Mf(t)\, F(t)\, dt$$

for all $F \in C(\mathbb{R})$. One can show that $M(D(X)) = \mathcal{H}$, where

$$\mathcal{H} = \left\{ \varphi + \eta_0 \varphi_0 + \eta_1 \varphi_1 \mid \varphi, \varphi_0, \varphi_1 \in D(\mathbb{R}) \right\}.$$

Here $\eta_0(t) = |t|^3 / 2$ and $\eta_1(t) = |t-1|^7 / 2$ $(t \in \mathbb{R})$. We normalize $\Box$ in the following way : $\Box\,(F\,.\,Q) = (LF)\,.\,Q$ for all $F \in C^2(\mathbb{R})$, where

$$(LF)(t) = 4t(t-1)\, F''(t) + 16(3t-1)\, F'(t) \qquad (t \in \mathbb{R}).$$

We are led to the differential equation :

$$L\, S = (s^2 - 121)\, S \qquad (\ast)$$

on $\mathbb{R}$, for complex s. Define some special solutions of ($\ast$) by :

$$W^s(t) = \begin{cases} {}_2F_1((11+s)/2,\ (11-s)/2;\ 8;\ 1-t) & (t>1) \\ 0 & (t \leq 1) \end{cases},$$

$$V^s(t) = \begin{cases} {}_2F_1((11+s)/2,\ (11-s)/2;\ 4;\ t) & (t<0) \\ 0 & (t \geq 0) \end{cases},$$

$$S^r(t) = {}_2F_1(-r,\ 11+r;\ 4;\ t) \qquad (r = 0,1,2,\ldots;\ t \in \mathbb{R}).$$

In the obvious way these functions are considered as elements of $\mathcal{H}'$. Now we examine the following conditions :

$$W(t) = t^{-3}\, W_1(t) + \alpha_0(s)\, \Phi(t) \log|t| \qquad (t \text{ near } 0,\ t \neq 0)$$

W satisfies ($\ast$)

Φ is an analytic solution of ($\ast$) near 0; $\Phi(0) = 1$

W_1 is analytic near 0; $W_1(0) = 1$

$$W_1(t) = \sum_{k=0}^{\infty} b_k(s)\, t^k \quad \text{near } 0$$

The numbers $b_k(s)$, with $k = 0,1,2$, and $\alpha_0(s)$ are uniquely determined. Notice that $b_0(s) = 1$. Similarly we consider the condition :

$$\tilde{W}(t) = (t-1)^{-7}\, \tilde{W}_1(t) + \alpha_1(s)\, \Psi(t) \log|t-1| \qquad (t \text{ near } 1)$$

The power series expansion of $\tilde{W}_1$ around 1 has coefficients $c_k(s)$ $(k = 0, 1,2,\ldots)$; $c_0(s),\ldots,c_6(s)$ and $\alpha_1(s)$ are uniquely determined. In fact all these numbers are easily computed. For complex s we let :

$$E_0^s(\varphi) = \sum_{k=0}^{2} b_{2-k}(s)\, \varphi^{(k)}(0) / \Gamma(k+1),$$

$$E_1^s(\varphi) = \sum_{k=0}^{6} c_{6-k}(s)\, \varphi^{(k)}(1) / \Gamma(k+1),$$

where $\varphi \in \mathcal{H}$.

Finally we have the following theorems :

<u>THEOREM 2</u> The H-orbit A carries a non-zero H-invariant measure, which can be viewed as a spherical distribution μ on X, with eigenvalue -96 ($= 5^2 - 121$).

<u>THEOREM 3</u> Let T be an H-invariant distribution on X. Then there exists a complex number c such that $T - c\mu = M'S$ for some $S \in \mathcal{H}'$. Here M' is the dual map from $\mathcal{H}'$ to D'(X).

From this, using our solutions of ($*$), we deduce :

<u>THEOREM 4</u> Let $\lambda = s^2 - 121$, $s \in \mathbb{C}$, Re $s \geqslant 0$. Write $B(s) = \left\{ M'(\alpha_1(s) W^s + E_1^s),\ M'(-\alpha_0(s) V^s + E_0^s) \right\}$. Then :

(a) If $s \notin \{5, 11, 13, 15, \dots\}$, then B(s) is a basis of $D'_{\lambda,H}(X)$.

(b) If $s = 11 + 2r$ with $r \in \{0, 1, 2, \dots\}$, then $\{M'S^r\} \cup B(s)$ is a basis of $D'_{\lambda,H}(X)$.

(c) $B(5) \cup \{\mu\}$ is a basis of $D'_{-96,H}(X)$.

Notice that the situation is quite similar to the one in case of $Sp(n, \mathbb{R}) / Sp(n-1, \mathbb{R}) \times Sp(1, \mathbb{R})$, except for the measure μ of course.

Finally we give the explicit formulas for $\alpha_j(s)$ $(j = 0,1)$:

$$\alpha_0(s) = -(s^2-25)(s^2-49)(s^2-81) / 768 ,$$

$$\alpha_1(s) = (s^2-1)^2(s^2-9)^2(s^2-25)(s^2-49)(s^2-81) / 2^{22}.3^4.5^2.7 .$$

4. THE PLANCHEREL FORMULA

Now that we have determined the spherical distributions, it is a matter of technique to compute the spectral decomposition of $\tilde{\Box}$. We first state the formula and then give some comments.

<u>THEOREM 5</u> The measure dx can be normalized in such a way (and this normalization can be explicitly described) that for all $f \in D(X)$:

$$\int_X | f(x) |^2 dx =$$

$$\frac{1}{2\pi} \int_{z=0}^{\infty} \frac{(z^2+1)(z^2+9)\Gamma((11+iz)/2)\Gamma((11-iz)/2)\ iz \sin(\pi iz/2)}{16\ \Gamma(8)} .$$

$$. M'\left(\frac{1}{\alpha_1(iz)} E_1^{iz} + W^{iz} \right) (f \# \overline{f})\, dz \qquad +$$

$$\sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(11+2r+1)}{6\Gamma(r+1)} M'\Big(S^r \Big|_{(0,1)} + \frac{1}{\alpha_0(11+2r)} E_0^{11+2r} \qquad +$$

$$\frac{(-1)^{r+1} \Gamma(4)\Gamma(r+8)}{\Gamma(8)\Gamma(r+4)\alpha_1(11+2r)} E_1^{11+2r} \Big) (f \# \overline{f}) \qquad +$$

$$M'\left(-720 \sum_{r=1}^{5} (11-2r)\, E_1^{11-2r} + 240\, E_0^{5} - 1680\, E_0^{7} + 6480\, E_0^{9} \right) (f * \bar{f}) .$$

Here $f * \bar{f}$ denotes the convolution product of f and $\bar{f}$, f being viewed as a function on G (in the usual way).

Finally we give some comments :

(a) The continuous spectrum should be examined. It will probably split into two parts, corresponding to certain irreducible unitary representations of G, induced from a parabolic subgroup canonically associated with H.

(b) The discrete spectrum corresponding to eigenvalues ≥ 0, consists of irreducible components. In fact, results of Oshima and Matsuki imply irreducibility even for eigenvalues ≥ -40. Also, Kengmana has obtained results in this direction.

(c) The most difficult part will be to decide (ir)reducibility for eigenvalues < 0. In particular -96 will provide serious problems. The others can probably be handled by using K-finite functions.

(d) As is easily proved, (G,H) is a so-called generalized Gelfand pair. Therefore we can speak of <u>the</u> Plancherel formula, once (a) and (c) are understood.

(e) Of course, the 'weight-functions' occurring in the Plancherel formula should be rewritten in terms of appropriate c-functions.