THE PLANCHEREL FORMULA FOR A SYMPLECTIC SYMMETRIC SPACE

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1. The setting

Let $G = Sp(n,\mathbb{R})$, the real $2n \times 2n$ matrices, preserving the standard symplectic form of $\mathbb{R}^{2n}$ (see [H]). Let E_{km} denote the $2n \times 2n$ matrix with exactly one nonzero entry, namely the $(k,m)^{th}$ element, which equals one. Define :

$$J = \sum_{k=1}^{n-1} (E_{kk} + E_{n+k,n+k}) - E_{nn} - E_{2n,2n} .$$

An involutive automorphism of G is given by $\sigma(g) = JgJ$ $(g \in G)$. The set of fixed points of σ is a connected Lie subgroup H of G; H is isomorphic to $Sp(n-1,\mathbb{R}) \times Sp(1,\mathbb{R})$. A Cartan-involution Θ of G, commuting with σ, is given by $\Theta(g) = {}^tg^{-1}$ $(g \in G)$. The set of fixed points of Θ is a maximal compact subgroup K of G, isomorphic to the reductive Lie group $U(n)$.

Define $X = G/H$. Now X is a semisimple symmetric space of split rank one and rank one. In this note we give a desintegration of $L^2(X)$ into subspaces corresponding to certain irreducible unitary representations of G. Especially the discrete part of the spectrum is interesting. The main result is given in Proposition 13; however, no proofs are given here : they will appear elsewhere. The aim of this note is to give an idea of the most important arguments and results.

The situation dealt with resembles the one discussed in [D,P] . Nevertheless, there exist several important differences. We shall return to this later. From now on we let $n > 2$, the case n=2 being easier.

2. Some structure theory

The Lie algebra of G is $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{R})$ (see [H]). The differentials of σ and Θ, again denoted by σ and Θ, are involutive automorphisms of $\mathfrak{g}$. It is easy to see that $\sigma(Y) = JYJ$ and $\theta(Y) = -{}^tY$ $(Y \in \mathfrak{g})$. The +1 and -1 eigenspaces of σ (resp. θ) are called $\mathfrak{h}$ and $\mathfrak{q}$ (resp. $\mathfrak{k}$ and $\mathfrak{p}$). Now $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{k}$ that of K. A maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p} \cap \mathfrak{q}$ (and of $\mathfrak{q}$) is given by $\mathfrak{a} = \mathbb{R}L$, where $L = E_{1,2n} + E_{n,n+1} + E_{n+1,n} + E_{2n,1}$. Let $A = \exp \mathfrak{a}$ and define

$$A(t) = \sinh^{2n-3} 2t \cosh^2 2t \qquad (t \in \mathbb{R}).$$

Let dk be the Haarmeasure on K, normalized by

$$\int_K dk = 1 .$$

Using the generalized Cartan-decomposition $G = KAH$ one can show that

$$\int_X \varphi(x)dx = \int_K \int_0^\infty (k \exp(tL)\, H)\, A(t)\, dt\, dk \quad (\varphi \in D(X))$$

defines a G-invariant measure on X. We write $L^2(X,dx) = L^2(X)$.

In order to construct a parabolic subgroup of G, we define

$$\mathfrak{g}_m = \{ Y \in \mathfrak{g} \mid ad(L)\, Y = mY \} , \qquad (m=1,2),$$

$$N = \exp(\mathfrak{g}_1 + \mathfrak{g}_2) ,$$

$$M = \{ h \in H \mid Ad(h)L = L \} ,$$

$$w = \sum_{k=2}^{n-1} (E_{kk} + E_{n+k,n+k}) + E_{1,2n} + E_{2n,1} - E_{n,n+1} - E_{n+1,n} .$$

Then $w \in K$ and w centralizes A. One computes that $\dim \mathfrak{g}_1 = m_1 = 4(n-2)$ and $\dim \mathfrak{g}_2 = m_2 = 3$; let $\rho = \frac{1}{2}m_1 + m_2 = 2n-1$. It is easy to check that $\overline{M}AN$ is a parabolic subgroup of G; here $\overline{M} = M \cup wM$. Define two characters χ_0 and χ_1 of $\overline{M}$ by $\chi_0 = 1$; $\chi_1(m) = -\chi_1(wm) = 1$ $(m \in M)$. These characters are used to define certain induced representations of G.

3. The model

Now we give a model for X, which is extremely useful for explicit computations; also the H-orbit structure on this model can be neatly described: see Lemma 2. Define, for k=0,2 :

$$X_k = \left\{ x = \begin{pmatrix} A & B \\ C & {}^tA \end{pmatrix} \;\middle|\; \begin{matrix} A,B,C \text{ real } n\times n \text{ matrices; } B,C \text{ skew} \\ \text{symmetric; rank } x = 2\text{; trace } x = k \end{matrix} \right\}$$

X_k is a manifold and G acts on it by $g\cdot x = gxg^{-1}$ ($x \in X_k$, $g \in G$). First we consider X_2; we shall return to X_0 later on. Let $x^0 = E_{n,n} + E_{2n,2n}$, an element of X_2. As is easily seen, the stabilizer of x^0 in G is equal to H and furthermore we have :

Lemma 1 X and X_2 are diffeomorphic.

From now on we identify X and X_2.

In order to give a description of the H-orbit structure on X, we introduce a real analytic function Q on X by $Q(x) = x_{nn}$ (x_{nn} denoting the (n,n) entry of the matrix x in X). Using the definition

$$X(c) = \left\{ x \in X \mid Q(x) = c \right\} \qquad (c \in \mathbb{R})$$

we have :

Lemma 2 (1) For $c \neq 0,1$ $X(c)$ is a $(4n-5)$ dimensional H-orbit.

(2) X(0) consists of two H-orbits : $H\cdot(E_{11} + E_{n1} + E_{n+1,n+1} + E_{n+1,2n}) = D_0$ and $X(0)-D_0$, of dimension 4n-5 and 4n-8 resp.

(3) X(1) consists of three H-orbits : $\left\{x^0\right\}$, $H\cdot(E_{11} + E_{nn} + E_{n+1,2n} + E_{2n,2n}) = D_1$ and $X(1)-D_1-\left\{x^0\right\}$, of dimension 0, 2n-1 and 4n-5 resp.

4. The mapping M

Loosely speaking, the mapping M gives the mean over the subsets X(c) of a function on X. Using M, certain differential operators on X can be viewed as linear differential operators on $\mathbb{R}$. The mapping M is defined as follows : for $\varphi \in D(X)$, $M\varphi$ is the function satisfying

$$\int_X F(Q(x))\, \varphi(x)\, dx = \int_{\mathbb{R}} F(t)\, M\varphi(t)\, dt$$

for all $F \in C_c(\mathbb{R})$. Let $\mathcal{H} = M(D(X))$. Then we have :

Lemma 3 $\mathcal{H}$ consists of all functions f of the form

$$f(t) = f_0(t) + f_1(t)\, Y(t)\, t + f_2(t)\, Y(t-1)\,(t-1)^{2n-3} \qquad (t \in \mathbb{R})$$

with $f_k \in D(\mathbb{R})$ $(k=0,1,2)$.

Here $Y(t) = 1$ for $t \geqslant 0$ and $Y(t) = 0$ for $t < 0$.

If $\mathcal{H}$ is equipped with a topology similar to the one described in [F], Appendix, then M becomes a continuous mapping. The dual M' of M maps $\mathcal{H}'$ into $D'(X)$. A crucial result, which is not true for all rank one spaces, is given in

Theorem 4 $M'(\mathcal{H}')$ consists of all H-invariant distributions.

For the proof of this theorem one can use the model, given before.

5. The spherical distributions

We shall determine the so-called spherical distributions, i.e. the H-invariant eigendistributions of the Laplace-Beltrami-operator on X. This differential operator $\square$ is, at least up to a scalar, determined by the Casimir-element of G. The scalar is implicitly given by

Lemma 5 If $F \in C^2(\mathbb{R})$, then $\square(F \circ Q) = (LF) \circ Q$, where L is the second order differential operator on $\mathbb{R}$ defined by

$$L = 4t(t-1)\,\frac{d^2}{dt^2} + 8(nt-1)\,\frac{d}{dt}\,.$$

$\square$ generates the ring of G-invariant differential operators on X.

We define for complex λ

$$D'_{\lambda,H}(X) = \left\{ T \in D'(X) \mid T \text{ H-invariant}, \square T = \lambda T \right\},$$

the space of spherical distributions. Using Theorem 4, we transfer these to elements S of $\mathcal{H}'$, satisfying $LS = \lambda S$. This leads to the following definitions, for $s \in \mathbb{C}$, $r \in \{0,1,2,\dots\}$ and $t \in \mathbb{R}$:

$$S^r(t) = {}_2F_1(\,-r\,,\ \rho+r\,;\ 2\,;\ t\,)$$

$$S_+^s(t) = \begin{cases} {}_2F_1(\,\tfrac12(\rho+s)\,,\ \tfrac12(\rho-s)\,;\ \rho-1\,;\ 1-t\,) & (t>1) \\ 0 & (t \leqslant 1) \end{cases}$$

$$\tilde{S}_-^s(t) = \begin{cases} {}_2F_1(\ \tfrac{1}{2}(\rho+s)\ ,\ \tfrac{1}{2}(\rho-s)\ ;\ 2\ ;\ t\ ) & (t<0) \\ 0 & (t\geq 0) \end{cases}$$

$$\alpha(s) = \frac{-\Gamma(\frac{1}{2}(\rho+s))\Gamma(\frac{1}{2}(\rho-s))}{\Gamma(\frac{1}{2}(s-\rho+4))\Gamma(\frac{1}{2}(-s-\rho+4))\Gamma(\rho-2)\Gamma(\rho-1)}$$

$$\tilde{\alpha}(s) = (\rho+s-2)\ (\rho-s-2)\ /\ 4\ .$$

Note that S^r is a Jacobi polynomial of degree r, $\tilde{\alpha}(\pm(\rho-2)) = 0$ and $\alpha(s) = 0$ for $s \in \{\pm 1, \pm 3, \ldots, \pm(\rho-2)\}$.

Using the Lebesgue measure dt on $\mathbb{R}$, S^r, S_+^s and S_-^s can be considered as elements of $\mathcal{H}'$. We also define two delta-like elements E^s and $\tilde{E}^s$ of $\mathcal{H}'$: let $f \in \mathcal{H}$, then

$$E^s(f) = f(0) \quad ,$$

$$\tilde{E}^s(f) = \sum_{k=0}^{2n-4} b_k(s)\ f^{(k)}(1)\ /\ \Gamma(k+1) \quad ,$$

$$b_k(s) = \frac{\Gamma(\rho-k-2)\Gamma(\frac{1}{2}(\rho-s-2))\Gamma(\frac{1}{2}(\rho+s-2))}{\Gamma(k+1)\Gamma(\rho-2)\Gamma(\frac{1}{2}(\rho-s-2k-2))\Gamma(\frac{1}{2}(\rho+s-2k-2))} \ ,$$

for $k \in \{0,1,2,\ldots,2n-4\}$; cf. [K,D], Appendix. After all these preparations we can state

Theorem 6 Let $\lambda = s^2-\rho^2$ with $s \in \mathbb{C}$, Re $s \geq 0$. Let us write $R(s) =$
$\{M'(\alpha(s)\ S_+^s + E^s),\ M'(-\tilde{\alpha}(s)\ \tilde{S}_-^s + \tilde{E}^s)\}$.

(1) If $s \notin \{\rho, \rho+2, \rho+4, \ldots\}$, then $D'_{\lambda,H}(X)$ is two-dimensional and has basis $R(s)$.

(2) If $s = \rho+2r$ (with $r \in \{0,1,2,\ldots\}$), then $D'_{\lambda,H}(X)$ is three-dimensional and has basis $R(s) \cup \{M'S^r\}$.

Observe that the dimensions differ from those for the spaces treated by [F] (dimension nearly always 1, sometimes 2) and by [K,D] (dimension always 2). Furthermore, for $s = \rho-2$ the basis consists of two delta-like distributions; this causes a strange phenomenon, discussed later on.

6. The representations

We give a construction for some representations of G, which are important for the Plancherel formula. For this purpose we use the set X_0, given before (see p. 4). It is perfectly possible to describe the representations entirely in terms of induced representations, but it is more convenient to have a model for G/MN at one's disposal.

Define $\Xi = X_0$ and $\xi^0 = E_{11} - E_{1,2n} - E_{nn} + E_{n,n+1} - E_{n+1,n} + E_{n+1,n+1} + E_{2n,1} - E_{2n,2n}$, an element of Ξ. Now Ξ is diffeomorphic to G/MN, MN being the stabilizer of ξ^0 in G. From now on, let $s \in \mathbb{C}$ and $j \in \{0,1\}$. Define

$$E_{j,s}(\Xi) = \left\{ \begin{array}{l} f \in C^\infty(\Xi) \mid f(gm\exp(tL)\cdot\xi^0) = \chi_j(m)e^{(s-\varrho)t}f(g\cdot\xi^0) \\ \text{for all } g \in G,\ m \in \overline{M} \text{ and } t \in \mathbb{R}. \end{array} \right\}$$

We have :

<u>Lemma 7</u> If $\xi \in \Xi$, then there are $k \in K$ and $t \in \mathbb{R}$ with $\xi = k\exp(tL)\cdot\xi^0$, t being unique and k being unique modulo $K \cap M$.

Using this lemma we can identify $E_{j,s}(\Xi)$and $C_j^\infty(B)$, where

$$B = \left\{ k\cdot\xi^0 \mid k \in K \right\} ,$$

$$C_j^\infty(B) = \left\{ f \in C^\infty(B) \mid f(-b) = (-1)^j f(b) \text{ for all } b \in B \right\}.$$

Of course, B is isomorphic to $K/(K\cap M)$. In this way $E_{j,s}(\Xi)$ inherits a topology.

Now we define the representation $\pi_{j,s}$ on the space $E_{j,s}(\Xi)$ by

$$(\pi_{j,s}(g)\,f)\,(\xi) = f(g^{-1}\cdot\xi) \quad (g \in G,\ f \in E_{j,s}(\Xi),\ \xi \in \Xi).$$

$\pi_{j,s}$ is a representation of G induced by a character of $\overline{M}AN$. Let $\pi'_{j,s}$ be the representation on $(E_{j,-s}(\Xi))'$ contragredient to $\pi_{j,-s}$.

For later use we mention a non-degenerate G-invariant bilinear form $\langle\cdot\,,\cdot\rangle$ on $E_{j,s}(\Xi)\times E_{j,-s}(\Xi)$, defined by

$$\langle f,h\rangle = \int_B f(b)\,h(b)\,db \quad (f \in E_{j,s}(\Xi),\ h \in E_{j,-s}(\Xi)).$$

Here the measure db on B is K-invariant and satisfies

$$\int_B db = 1\ .$$

7. Construction of $\zeta_{j,s}$

We construct certain spherical distributions $\zeta_{j,s}$, intimately connected with the representations $\pi_{j,s}$ just now defined.

For $x \in X$ and $\xi \in \Xi$, write $(x,\xi) = -\frac{1}{2}\,\mathrm{trace}(x\xi)$. Define for $s \in \mathbb{C}$, Re $s > \rho$, $f \in E_{j,-s}(\Xi)$, $j \in \{0,1\}$:

$$u_{j,s}(f) = \Gamma\left(\tfrac{1}{2}(s-\rho+2+2j)\right)^{-1} \int_B |(x^0,b)|^{\frac{1}{2}(s-\rho)} (\mathrm{sgn}(x^0,b))^j \, f(b)\, db \, .$$

Lemma 8 Fix f and j. Then the function $s \mapsto u_{j,s}(f)$ is analytic on $\{s \in \mathbb{C} \mid \mathrm{Re}\ s > \rho\}$ and can be analytically extended to $\mathbb{C}$. Denoting this extension again by $u_{j,\cdot}(f)$, we have that $u_{j,s}$ is an H-invariant element of $(E_{j,-s}(\Xi))'$ for all $s \in \mathbb{C}$.

Now we come to the definition of $\zeta_{j,s}$: let $j \in \{0,1\}$ and $s \in \mathbb{C}$, then

$$\zeta_{j,s}(\varphi) = \langle \pi'_{j,s}(\varphi)\, u_{j,s} \, , \, u_{j,-s} \rangle \qquad (\varphi \in D(G)).$$

We have :

Theorem 9 For $j \in \{0,1\}$ and $s \in \mathbb{C}$, $\zeta_{j,s}$ is in $D'_{\lambda,H}(X)$ with $\lambda = s^2 - \rho^2$. So $\zeta_{j,s}$ is a spherical distribution.

In a similar way one can define the Fourier transforms and the intertwining operators (cf. [F] and [K,D]). However, they are not necessary for stating our results and therefore we shall omit the definitions here.

8. Asymptotics

In order to belong to the relative discrete series of X (i.e. occur as a discrete summand in the Plancherel formula), it is necessary for a spherical distribution to have a certain asymptotic behaviour, cf. [D,P] and [K]. More precisely, we have :

Theorem 10 Let π be a unitary representation of G on a Hilbert subspace of $L^2(X)$. Let T be its 'reproducing distribution' and assume that T is in $D'_{\lambda,H}(X)$ for a certain $\lambda = s^2-\rho^2$, with $s \in \mathbb{C}$ and Re $s \geq 0$. Then $s \in \{1,3,5,\dots\}$ and

(1) if $s \in \{1,3,\dots,\rho-4\}$, then T is a multiple of $M'E^s$,

(2) if $s = \rho-2$, then T is a linear combination of $M'E^{\rho-2}$ and $M'\tilde{E}^{\rho-2}$,

(3) if $s = \rho+2r$ (with $r \in \{0,1,2,\dots\}$), then T is a multiple of $M' \left(S^r \Big|_{(0,1)} + \right.$

$$\left. \frac{-1}{(\rho+r-1)\,(r+1)} \tilde{E}^{\rho+2r} + (-1)^{r+1} \frac{\Gamma(\rho-2)\Gamma(r+1)}{\Gamma(\rho+r)} E^{\rho+2r} \right).$$

Furthermore, using intertwining operators one can express $\zeta_{j,s}$ $(j \in \{0,1\}$, $s \in \mathbb{C})$ in the basis elements given in Theorem 6. For this purpose it appears to be sufficient to consider two special K-types. We take the trivial one and the K-type corresponding to a representation τ of K, that contains a unique vector v which is $K\cap M$-fixed and satisfies $\tau(w)\, v = -v$. Such a representation of U(n) (which is isomorphic to K) is given by (τ, V), where V is the space of complex skew symmetric $n \times n$ matrices and where U(n) acts by $\tau(k)\, Y = k\, Y\, {}^t k$ $(k \in U(n),\ Y \in V)$. For these two K-types very explicit computations are possible, cf. [K,D]. Instead of giving all numerical results, we state

<u>Theorem 11</u> Let $s \in \mathbb{C}$ with Re $s \geq 0$.

(1) If $s \in \{1,3,5,\dots\}$, $\zeta_{0,s}$ and $\zeta_{1,s}$ are linearly independent.

(2) If $s \in \{1,3,\dots,\rho-4\}$ $\zeta_{0,s}$ and $\zeta_{1,s}$ are nonzero, but coincide up to a scalar (see p.12, comment 2).

(3) $\zeta_{0,\rho-2} = 0$. However, $\zeta_{1,\rho-2}$ and $(d/dt)\big|_{t=\rho-2}\zeta_{0,t}$ are linearly independent (notice that this last distribution is in $D'_{\lambda,H}(X)$, with $\lambda = (\rho-2)^2-\rho^2$)

(4) Let $s \in \{\rho,\rho+2,\dots\}$; then $\zeta_{0,s} = \zeta_{1,s} = 0$, but

$(d/dt)|_{t=s}\zeta_{0,t}$ and $(d/dt)|_{t=s}\zeta_{1,t}$ are linearly independent.

For example, one shows :

$$(d/dt)|_{t=\rho-2}\zeta_{0,t} = \tfrac{1}{4}\Gamma(n)\Gamma(n-1)\,(-1)^{n+1}\,M'(E^{\rho-2}+\tilde{E}^{\rho-2})\ ,$$
$$\zeta_{1,\rho-2} = \tfrac{1}{2}\Gamma(n)\Gamma(n-1)\,(-1)^{n}\,(\rho-2)^{-1}\,M'(E^{\rho-2}-\tilde{E}^{\rho-2})\ .$$

These distributions will play an important role in what follows.

9. The c-functions

We shall define the c-functions $c_j(s)$ $(j=0,1)$, the analogues of Harish-Chandra's c-function. The c-functions occur in the asymptotic behaviour of the distributions $\zeta_{j,s}$ (cf. [F], p.409) and can be explicitly computed :

$$c_0(s) = 2^{\rho-2}\Gamma(n)\Gamma(n-1)\pi^{-\frac{1}{2}}\Gamma(\tfrac{1}{2}s)\,/\,\Gamma(\tfrac{1}{2}(s+1))\Gamma(\tfrac{1}{2}(s+\rho-2))\ ,$$
$$c_1(s) = -2\,c_0(s)\,/\,(s+\rho-2) \qquad (s\in\mathbb{C})\ .$$

These functions are determined up to a scalar, coming from the choice of the measure dx on X. However, more important for the Plancherel formula is the set of poles of $1/c_j(s)c_j(-s)$ $(j=0,1)$, and for this set the scalar does not matter. One easily checks :

<u>Lemma 12</u> The poles of the functions $1/c_j(s)c_j(-s)$ are situated in the odd integers $(j=0,1)$. Both $1/c_0(s)c_0(-s)$ and $1/c_1(s)c_1(-s)$ have first order poles in $\{\pm1,\pm3,\dots,\pm(\rho-4)\}$ and second order poles in $\{\pm\rho,\pm(\rho+2),\dots\}$. Furthermore, $1/c_0(s)c_0(-s)$ has a second order pole in $\rho-2$, whereas $1/c_1(s)c_1(-s)$ has a first order pole in $\rho-2$.

10. The Plancherel formula

Before stating the main result of this note, the Plancherel formula for the space X, we introduce some notations. Define

$$c = 2^{2\varrho-3}\Gamma(n)\Gamma(n-1)\pi^{1-\varrho} \quad .$$

Define $\varphi_1 \# \varphi_2$ (for φ_1, φ_2 in $D(X)$) as in [F], p.425. Let $a_{-1}(f(s),s_0)$ denote the residue of the meromorphic function $f(s)$ in s_0 and write $a_{-2}(f(s),s_0) = a_{-1}(f(s)(s-s_0),s_0)$. Let $r \in \{0,1,2,\dots\}$. We define the spherical distribution Θ_r for even r by

$$\Theta_r = (d/dt)|_{t=\varrho+2r}\zeta_{0,t} + \tfrac{1}{4}(-1)^{n+1}\Gamma(n)\Gamma(n-1)\ (r+1)(\varrho+r-1)\ M'S^r$$

and for odd r by

$$\Theta_r = (d/dt)|_{t=\varrho+2r}\zeta_{1,t} + \tfrac{1}{4}(-1)^{n}\Gamma(n)\Gamma(n-1)\ M'S^r \quad .$$

These distributions are nonzero multiples of those mentioned in Theorem 10 ($\mathcal{S}$), cf. [F], chapter 8.

Now we can give the Plancherel formula for the space X :

<u>Proposition 13</u> Let $\varphi \in D(X)$. Then :

$$c^{-1}\int_X |\varphi(x)|^2\, dx =$$

$$\frac{1}{2\pi}\int_{\nu=0}^{\infty}\left\{\frac{\zeta_{0,i\nu}(\varphi\#\overline{\varphi})}{c_0(i\nu)c_0(-i\nu)} + \frac{\zeta_{1,i\nu}(\varphi\#\overline{\varphi})}{c_1(i\nu)c_1(-i\nu)}\right\} d\nu \ +$$

$$\sum_{r=-n+1}^{-1} a_{-1}\left(\frac{1}{c_1(s)c_1(-s)}, \varrho+2r\right)\zeta_{1,\varrho+2r}(\varphi\#\overline{\varphi}) \ +$$

$$\sum_{\substack{r=0\\ r \text{ even}}}^{\infty} a_{-2}\left(\frac{1}{c_0(s)c_0(-s)}, \varrho+2r\right)\Theta_r(\varphi\#\overline{\varphi}) \ +$$

$$\sum_{\substack{r=1\\ r \text{ odd}}}^{\infty} a_{-2}\left(\frac{1}{c_1(s)c_1(-s)}, \varrho+2r\right)\Theta_r(\varphi\#\overline{\varphi}) \quad .$$

That this is the Plancherel formula for X means among other things, that all distributions occurring in the right hand side are extremal in the cone of positive definite spherical distributions with the corresponding eigenvalue. So they are the 'reproducing distributions' of irreducible unitary representations of G, except for -possibly- the distributions $\zeta_{j,0}$ $(j=0,1)$.

11. Some comments

1. The proof of the formula itself is along the same lines as that in [F], p.424-428. An important step is the description of the H-invariant distributions T on X that satisfy the equation $\square T - \lambda T = \delta(x^0)$ ($\lambda \in \mathbb{C}$; $\delta(x^0)$ is the Dirac δ-function at x^0).

2. One can show, that for $r \in \{-n+1,-n+2,\ldots,-2\}$:

$$a_{-1}(\ 1/c_0(s)c_0(-s)\ ,\ \rho+2r\ )\zeta_{0,\rho+2r} =$$
$$a_{-1}(\ 1/c_1(s)c_1(-s)\ ,\ \rho+2r\ )\zeta_{1,\rho+2r}\ .$$

3. Theorem 10 is crucial for the proof of Proposition 13. One sees that all distributions mentioned there indeed occur in the Plancherel formula, except for $M'(E^{\rho-2} + \tilde{E}^{\rho-2})$. In fact, using K-fixed functions, one can show that the distribution $M'(E^{\rho-2} - \tilde{E}^{\rho-2})$, occurring in the Plancherel formula (it is a multiple of $\zeta_{1,\rho-2}$), is extremal. Here the situation is quite different from those described in [F] and [K,D].

4. It would be interesting to know the complete set of positive definite spherical distributions. In particular, I do not know whether $M'(E^{\rho-2} + \tilde{E}^{\rho-2})$ is positive definite or not.

5. The reader should compare Proposition 13 with the results of Oshima, see for instance [O].

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