TILES

from pattern to computation

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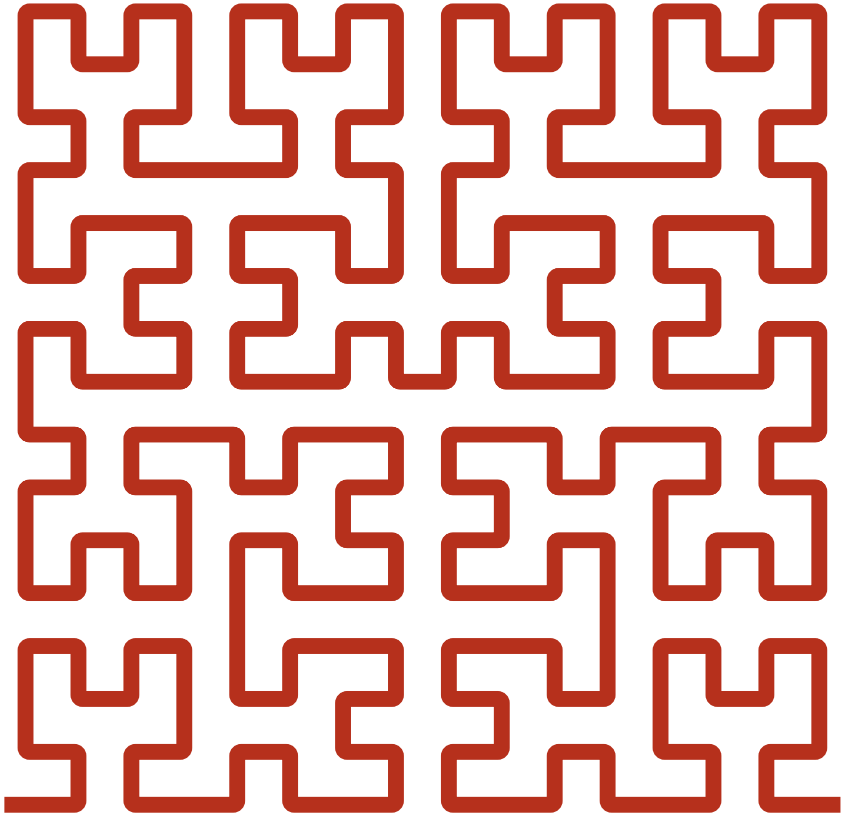
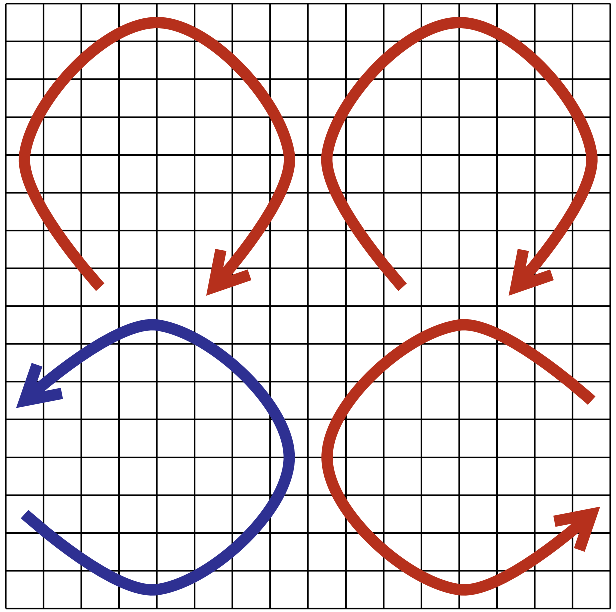
■ personal motivation



- 1-dim line
- 2-dim pattern

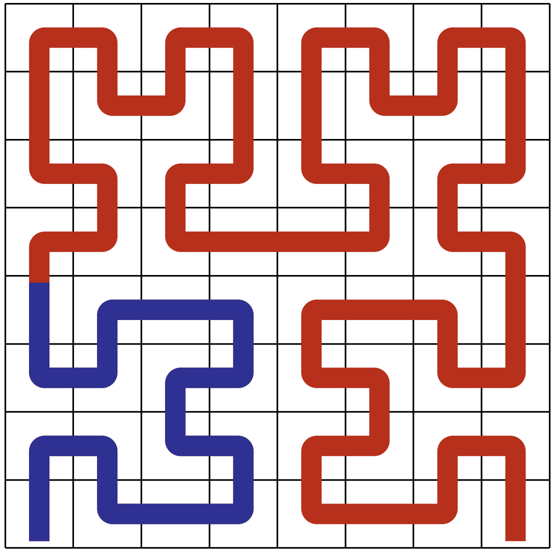


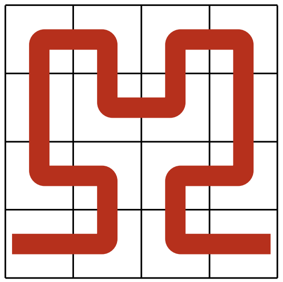
wikipedia

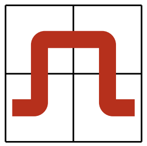
```
to starthilbert :size :level
  hilbert (:size / power 2 (:level-1)) :level 1
end

to hilbert :size :level :parity
  if  :level = 0 [stop]
  right 90 * :parity
  hilbert :size (:level-1) (:parity * -1)
  forward :size
  right -90 * :parity
  hilbert :size (:level-1) :parity
  forward :size
  hilbert :size (:level-1) :parity
  right -90 * :parity
  forward :size
  hilbert :size (:level-1) (:parity * -1)
  right 90 * :parity
end
```

call `starthilbert 200 5`

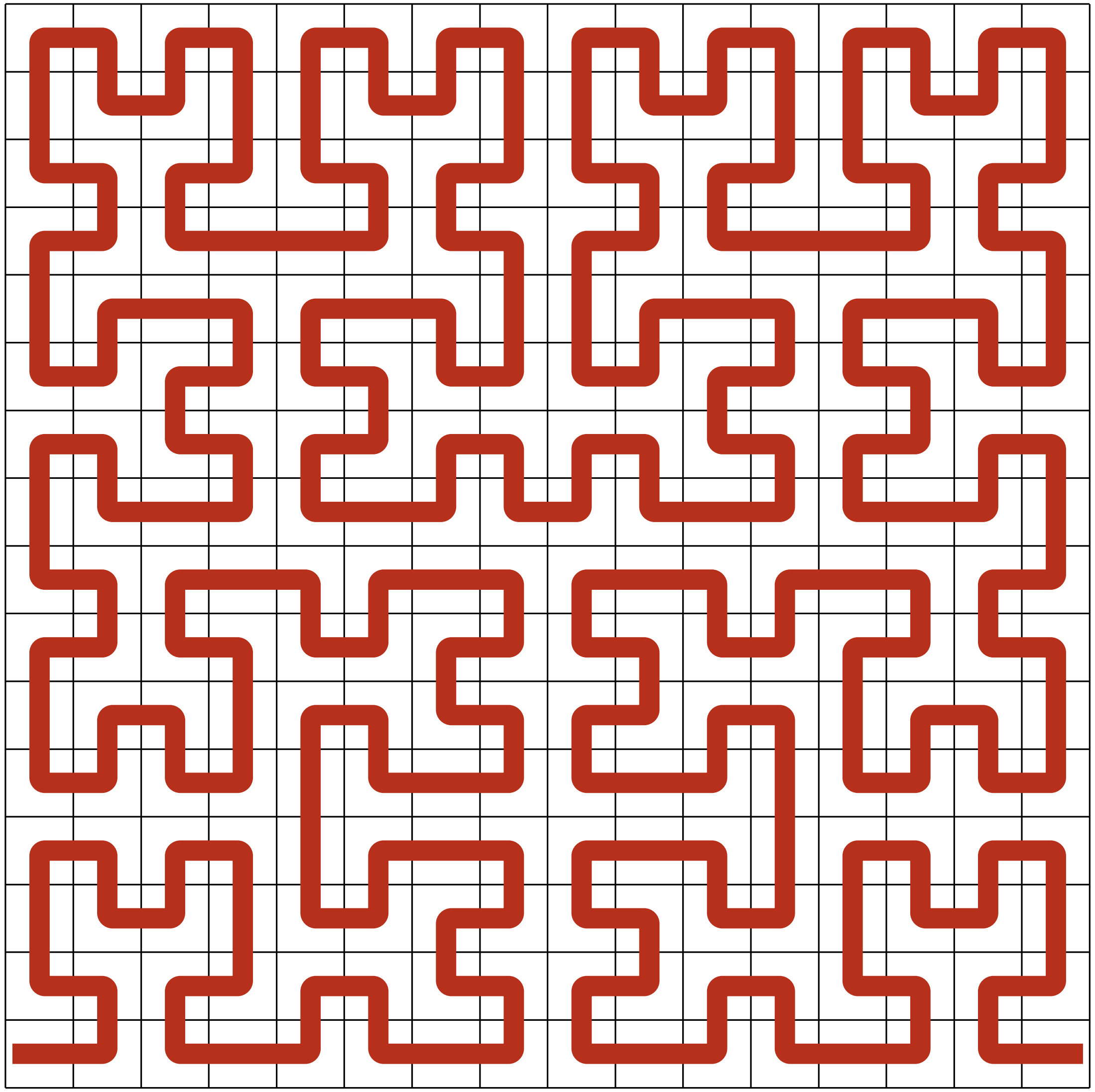




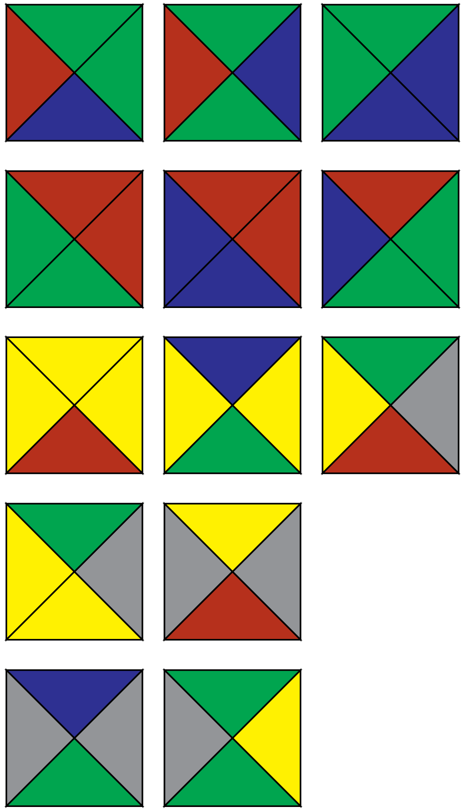


six ‘tiles’:

two lines, four curves



■ background



aperiodic tiling pattern (hard)

Karel Culik II, 1996

input: set of tiles

question: do they tile the plane?

(a rectangle)

matching edges, no rotations

1961 Wang decidable

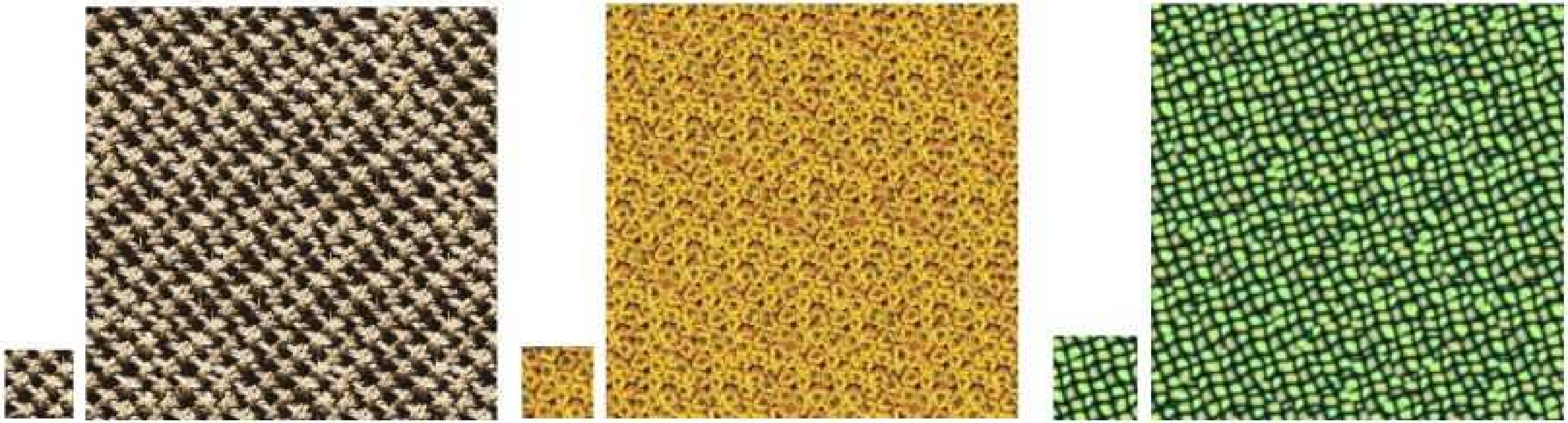
assumption: tiling is periodical

1966 Berger undecidable

aperiodic tiling (with 20,426 tiles)

1996 Culik 13 tiles

1974 Penrose 2 shapes (kite & dart)



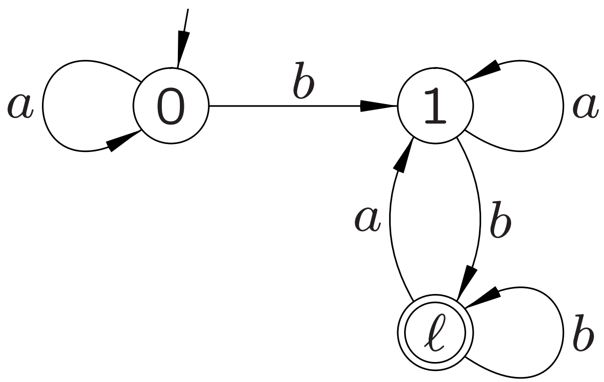
Cohen, Shade, Hiller, and Deussen: Wang Tiles for Image and Texture Generation. ACM Transactions on Graphics 22 (2003) 287-294.

Ares Lagae: Wang Tiles in Computer Graphics, 2009

aim: picture specifications

two-dimensional theory

strings: Chomsky hierarchy
grammars, automata, logic, . . .

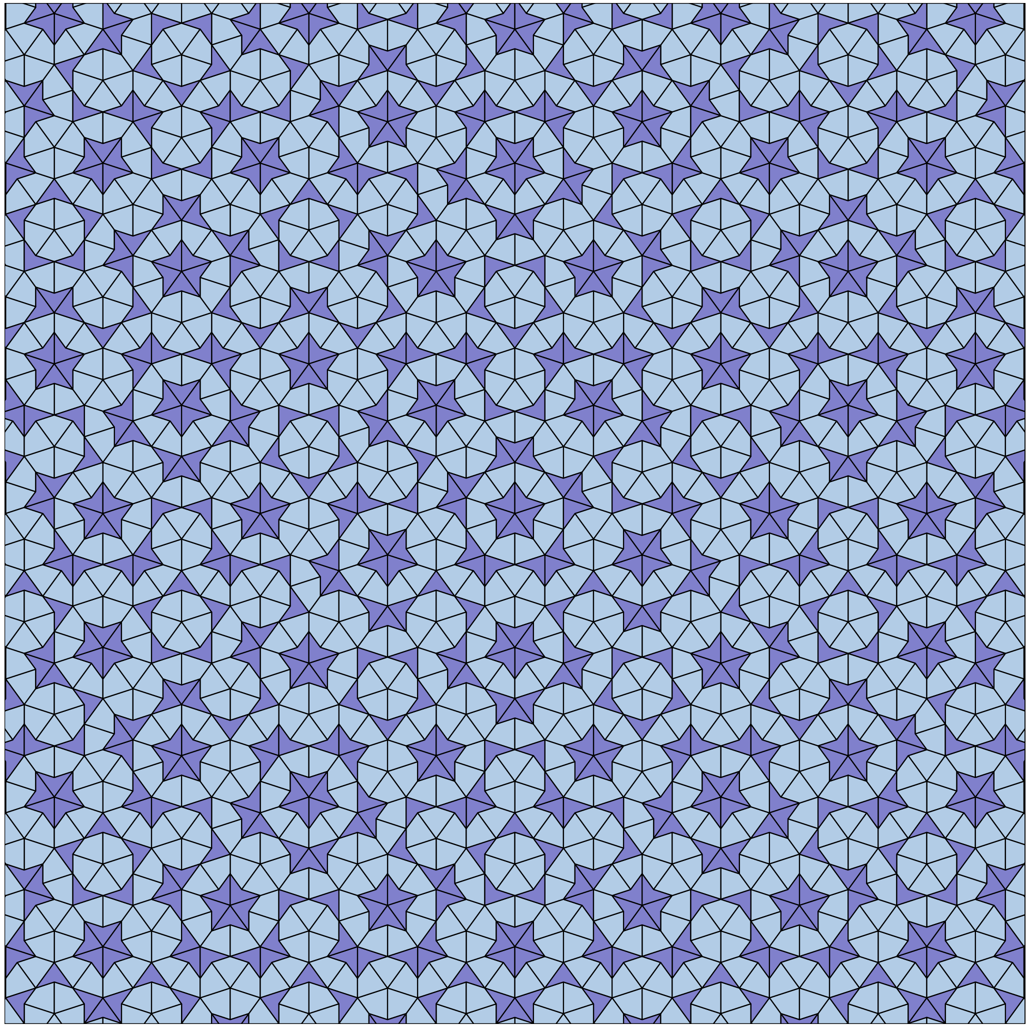


finite state specification

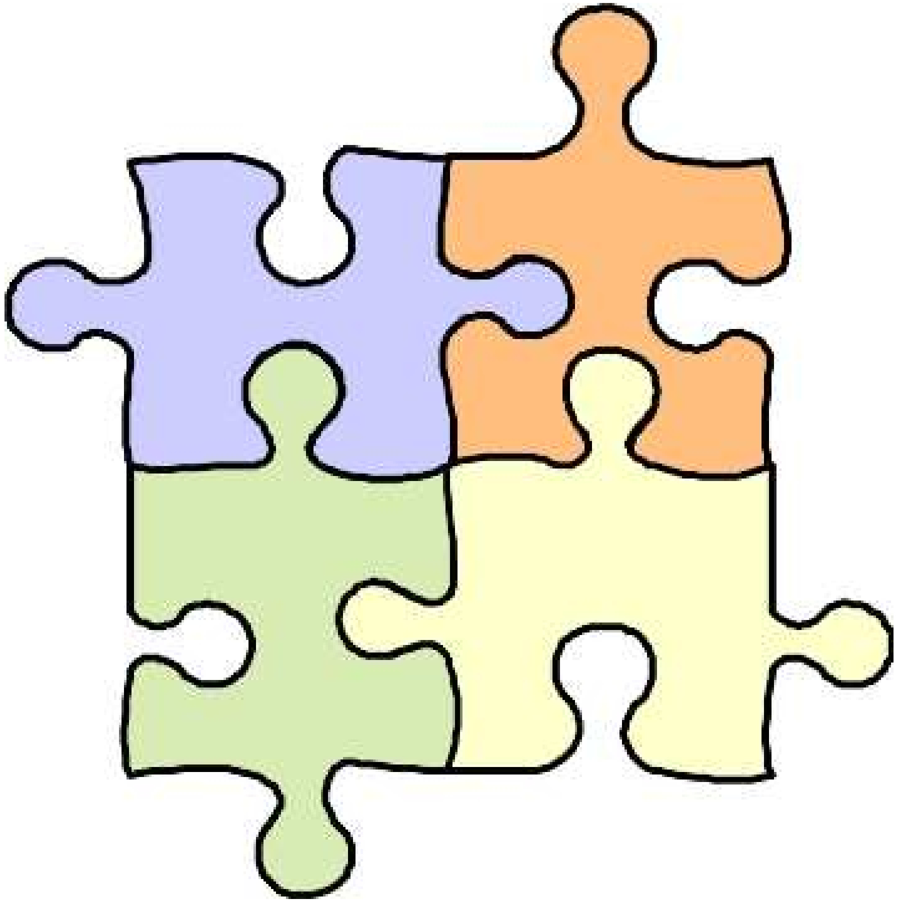
FSA

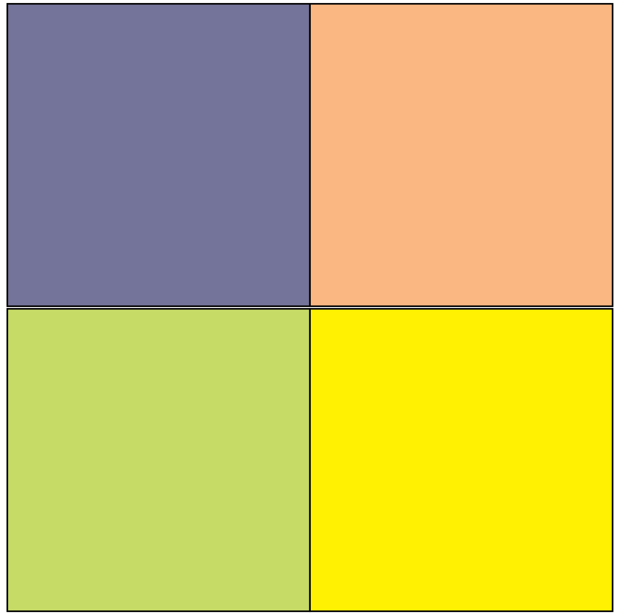
- (non-)determinism
- closure properties
  union, intersection, iteration, complement, . . .
- decision properties
  emptiness, inclusion, . . .
- regular languages
  automata, expressions, grammars, logic. . .

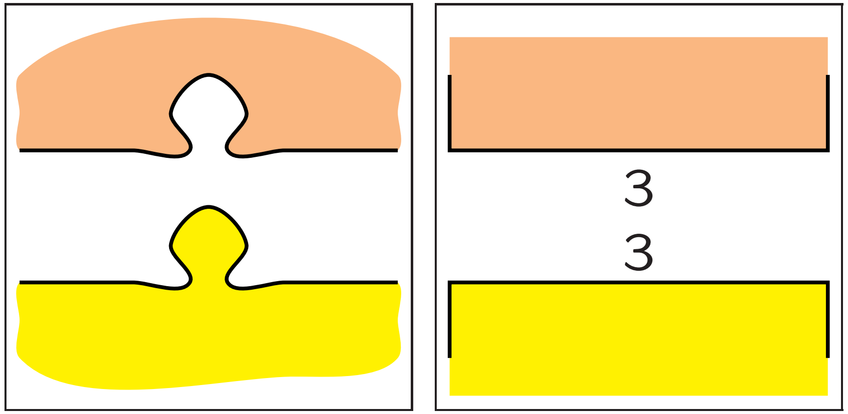
■ Rules of the game

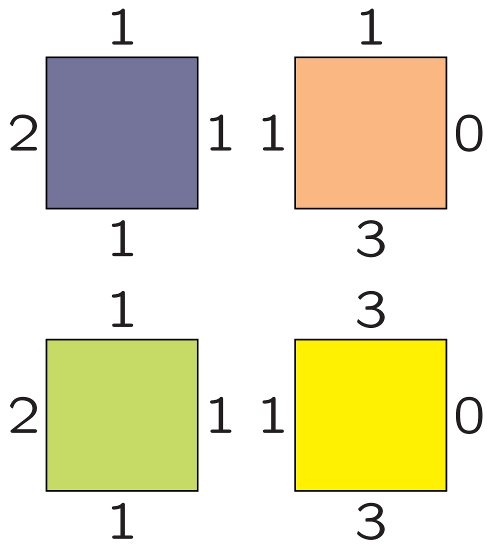


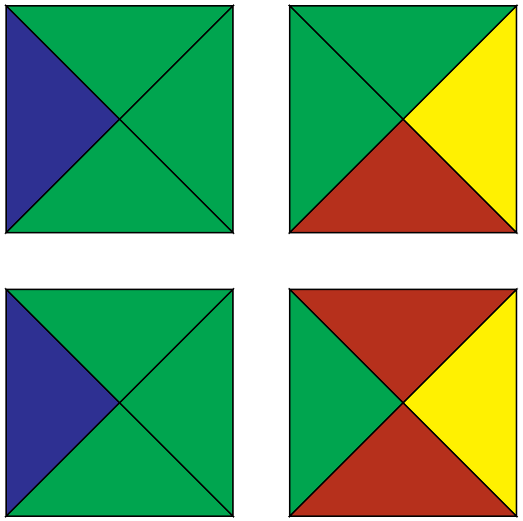
Penrose (1974)

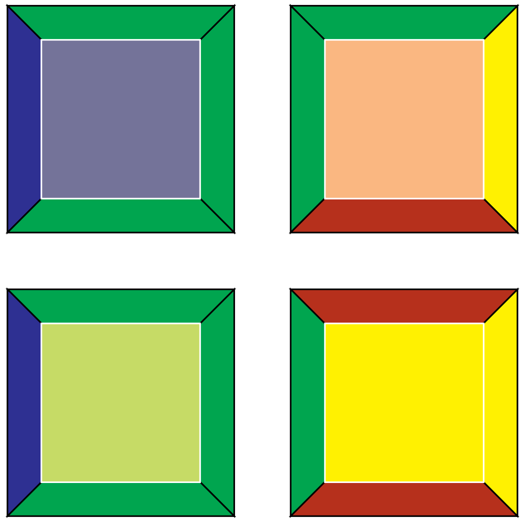


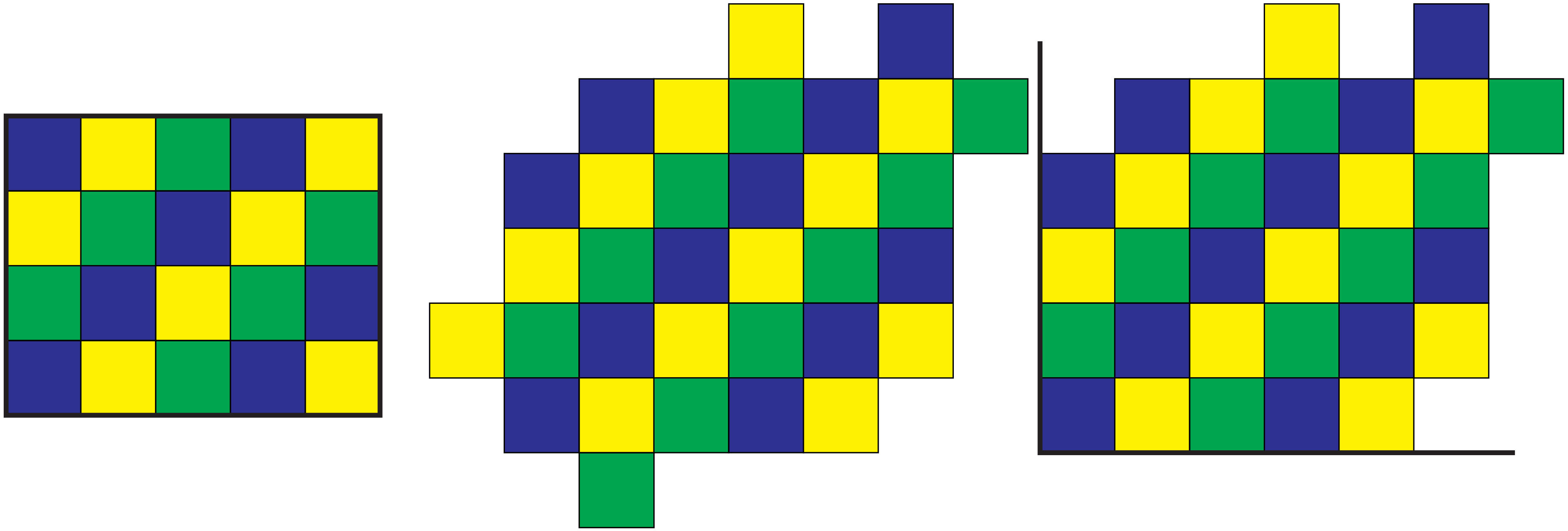



3
3


1 1
2 1 1 0
1 3
1 3
2 1 1 0
1 3

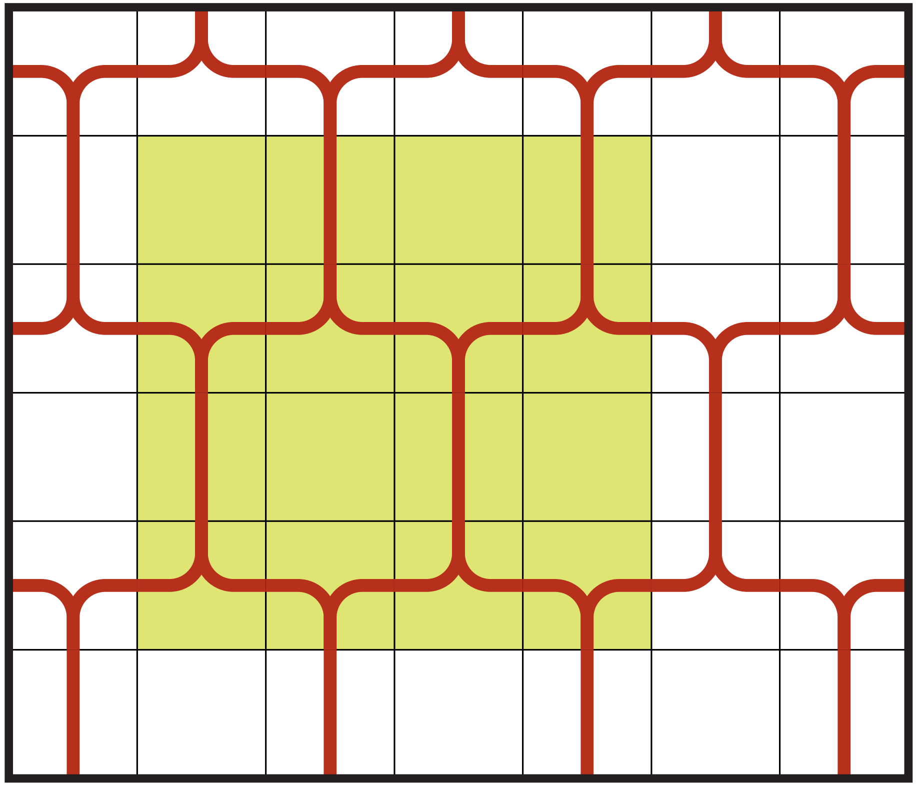


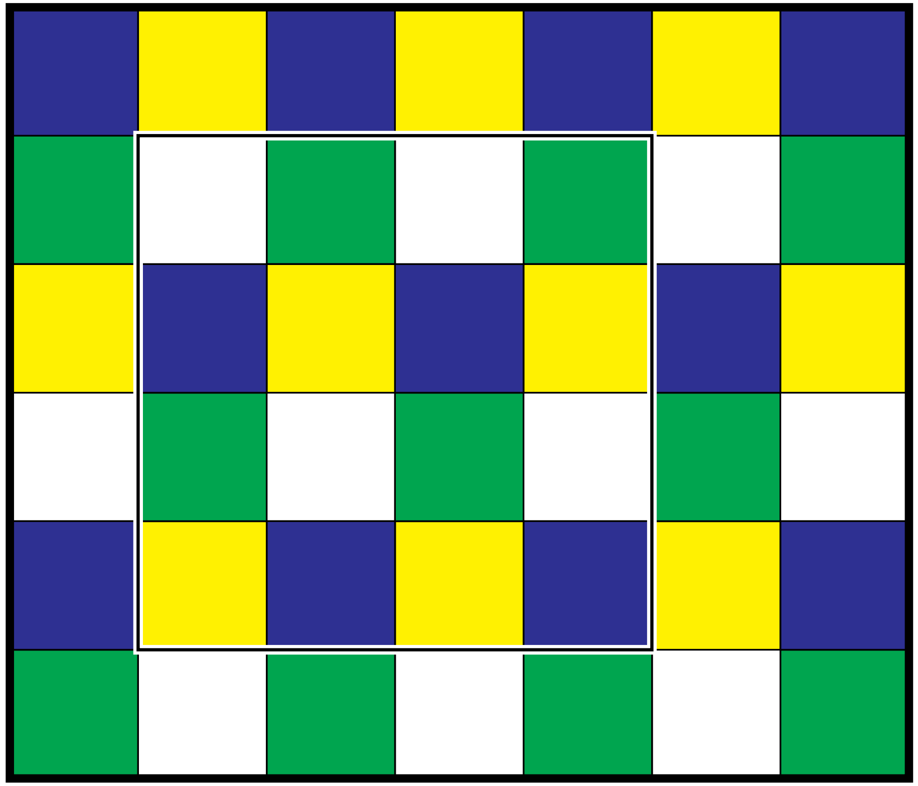




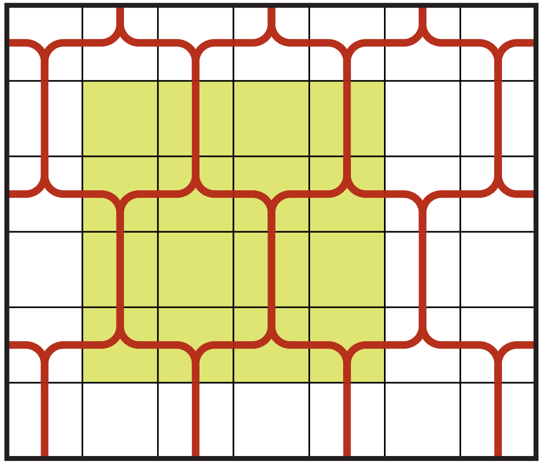
- rectangle
- (half) plane
- quadrant

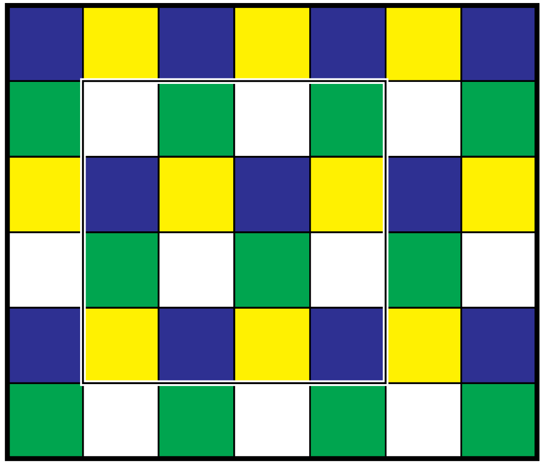
■ Examples

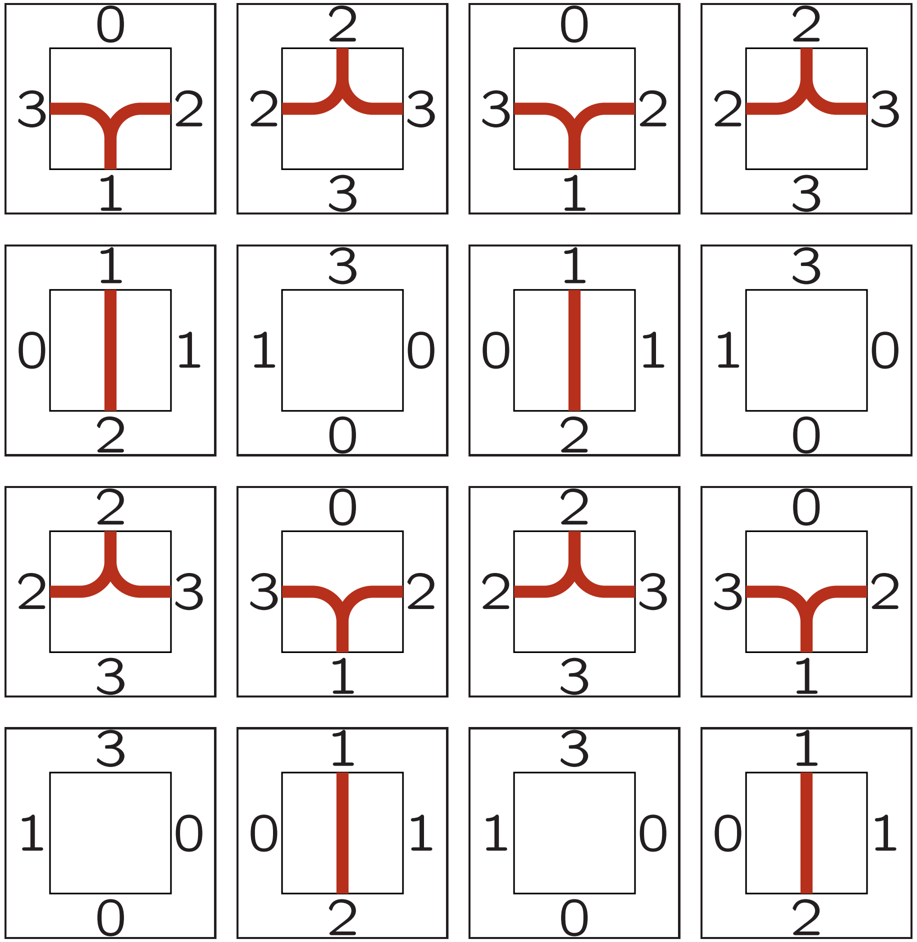




four ‘colours’

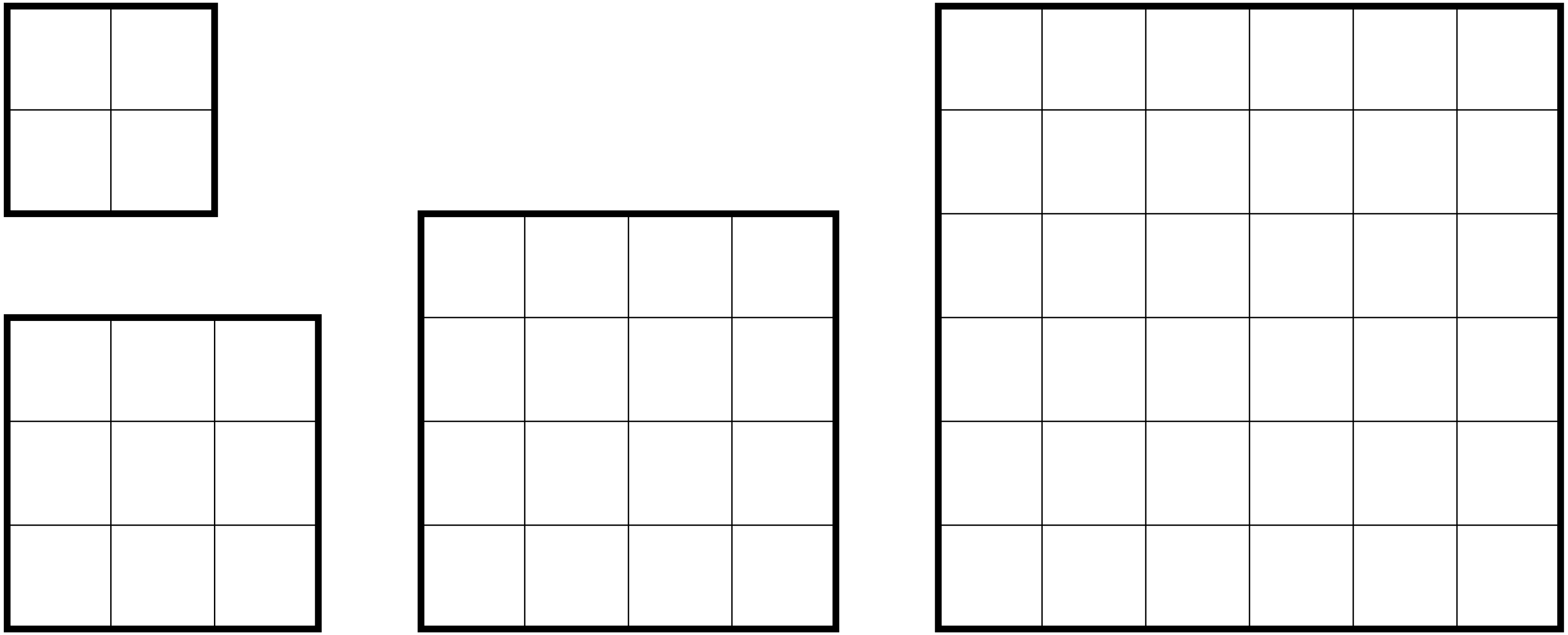




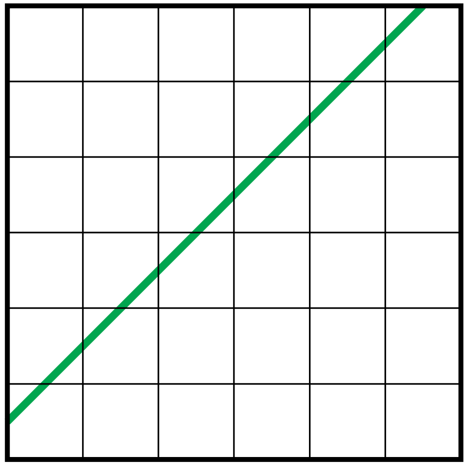


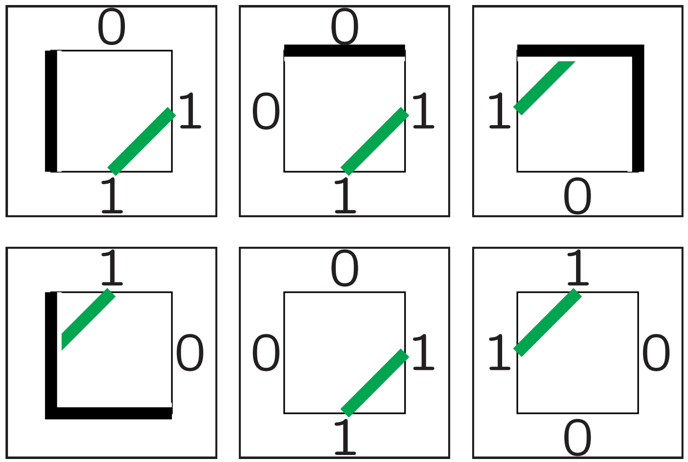
four tiles

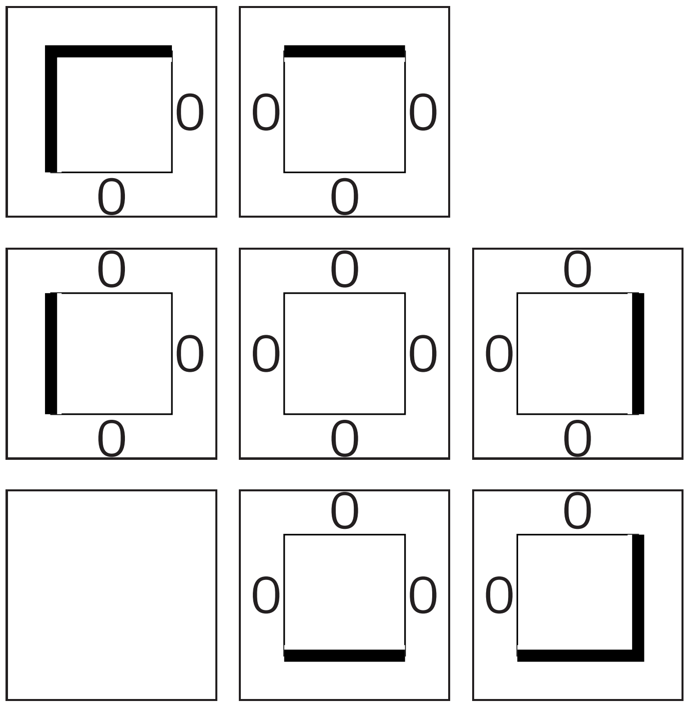
(and some borders)

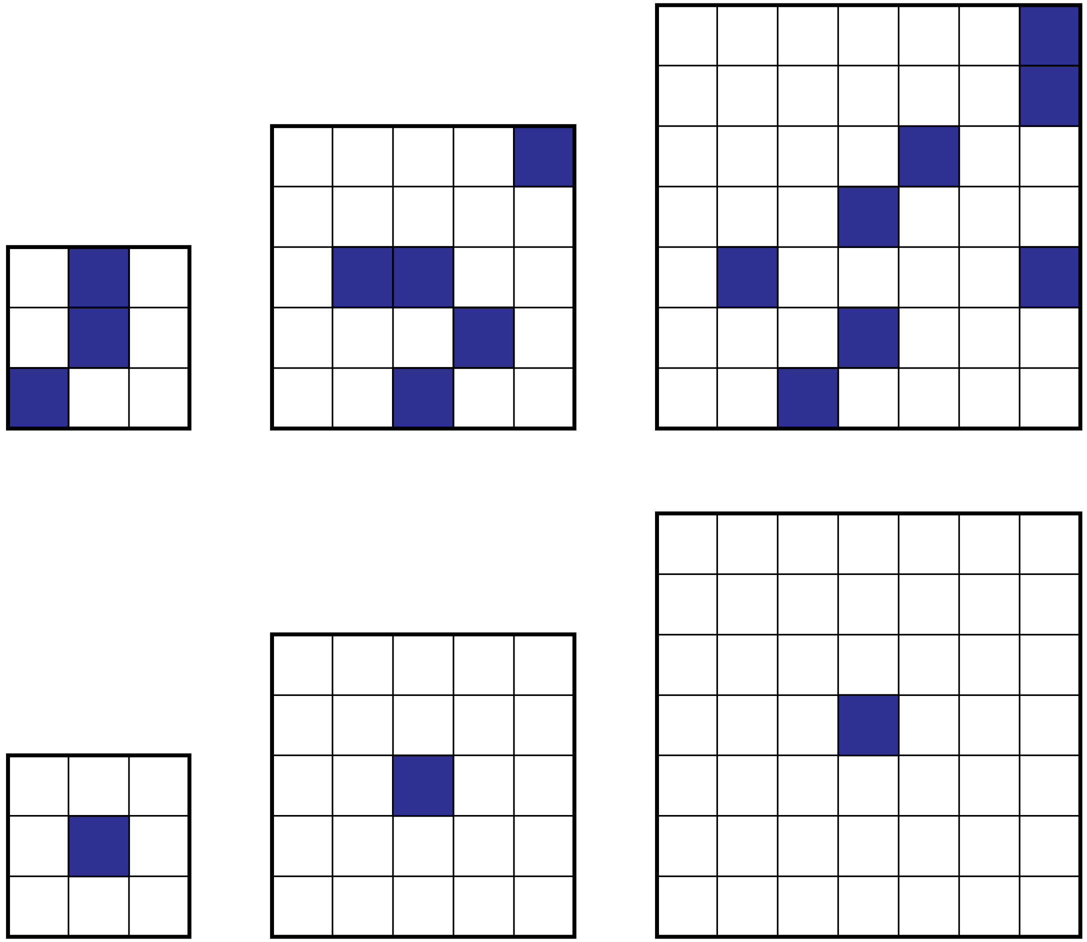


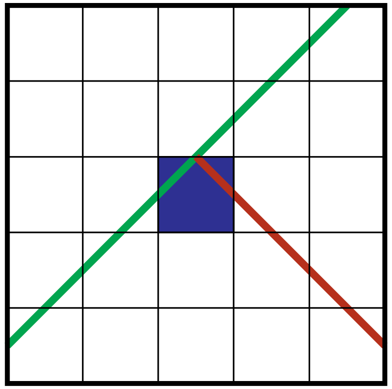
etcetera . . .

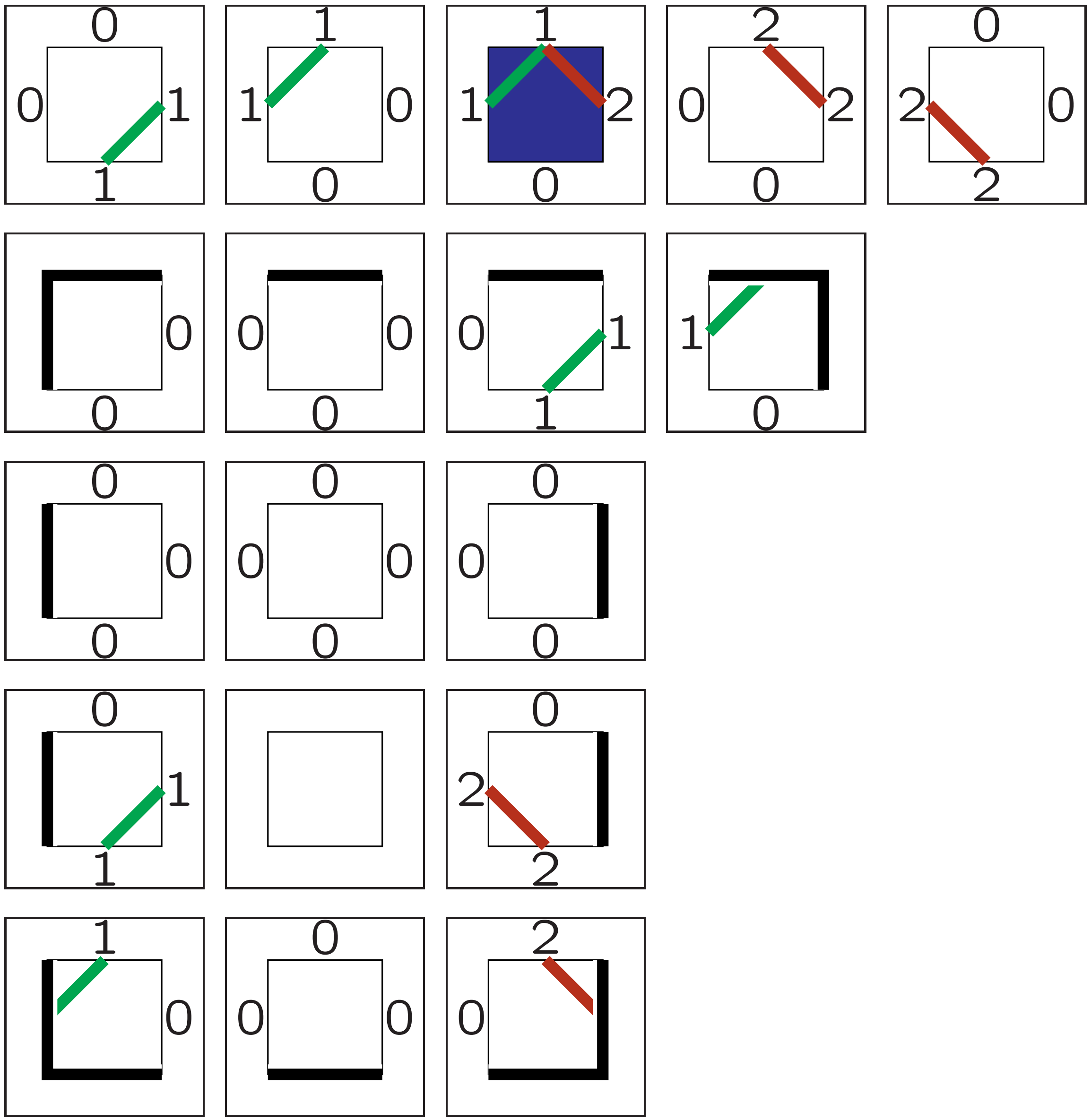


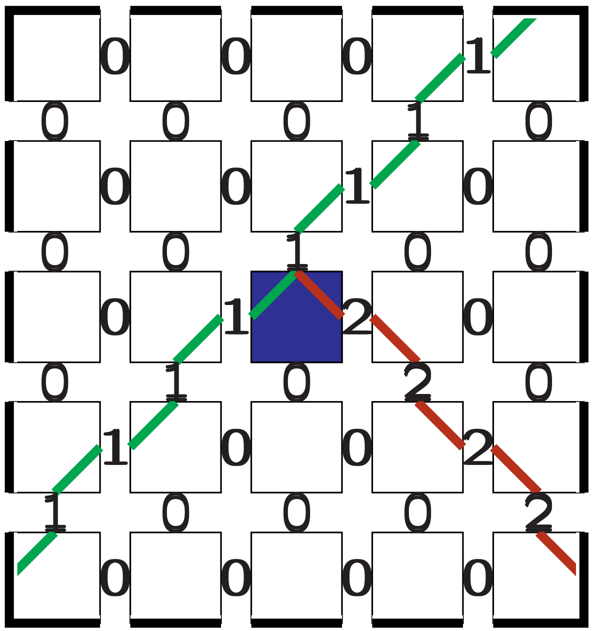


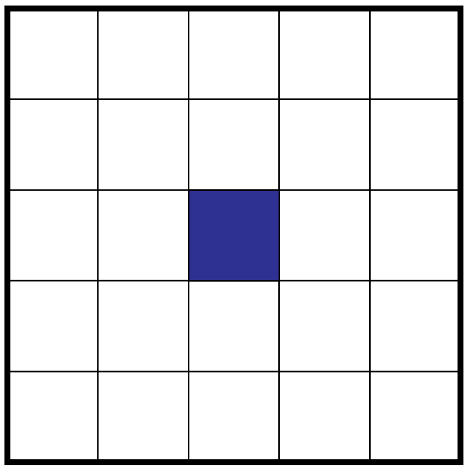












tiling system:

$(\Sigma, \Gamma, T, c, \varphi)$

Σ, Γ tile and edge colours

T tiles with four-sided markings

$c \in \Gamma$ border marking

$\varphi : T \to \Sigma$ tile $\mapsto$ colour

tiling: markings (in Γ) match

Wang picture language

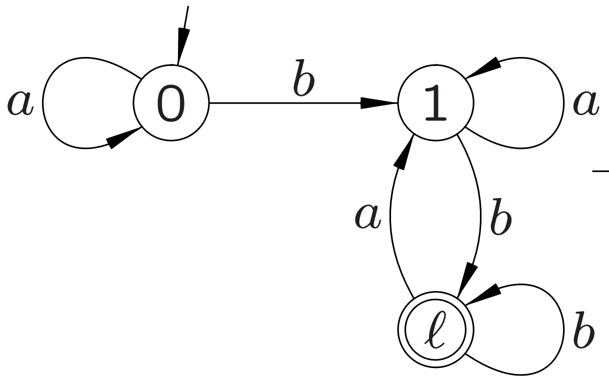
Giammarresi, Restivo:

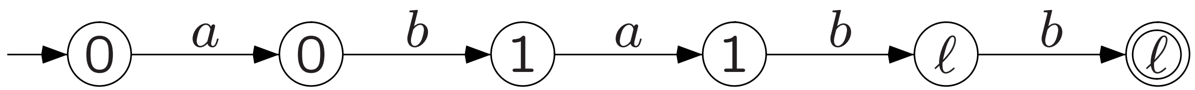
REC – *recognizable*

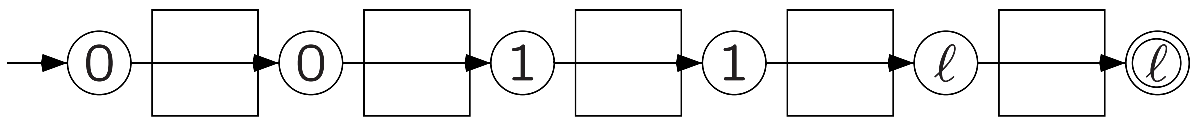
local lattice languages

h(LLL)

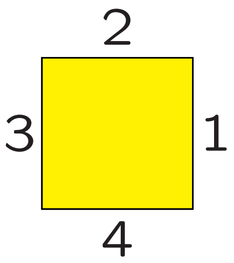
computation: finding transitions

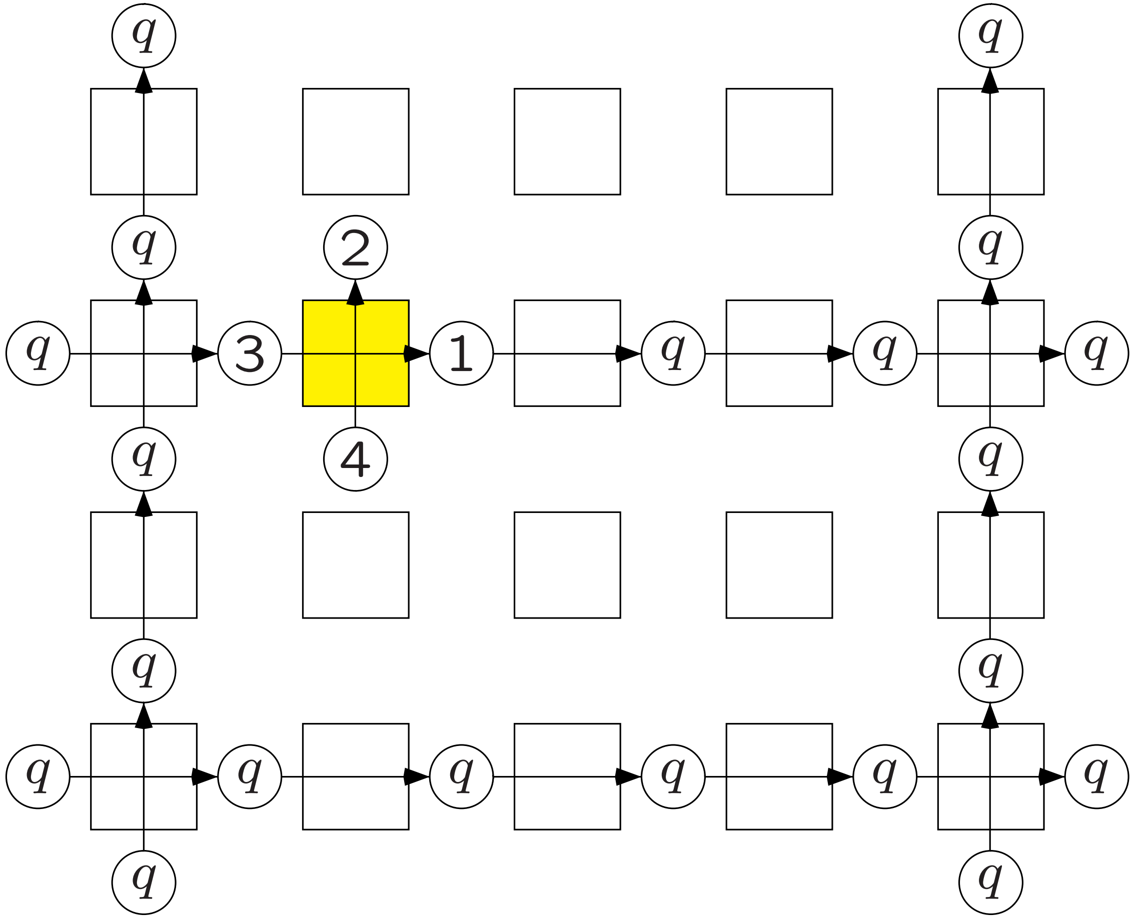




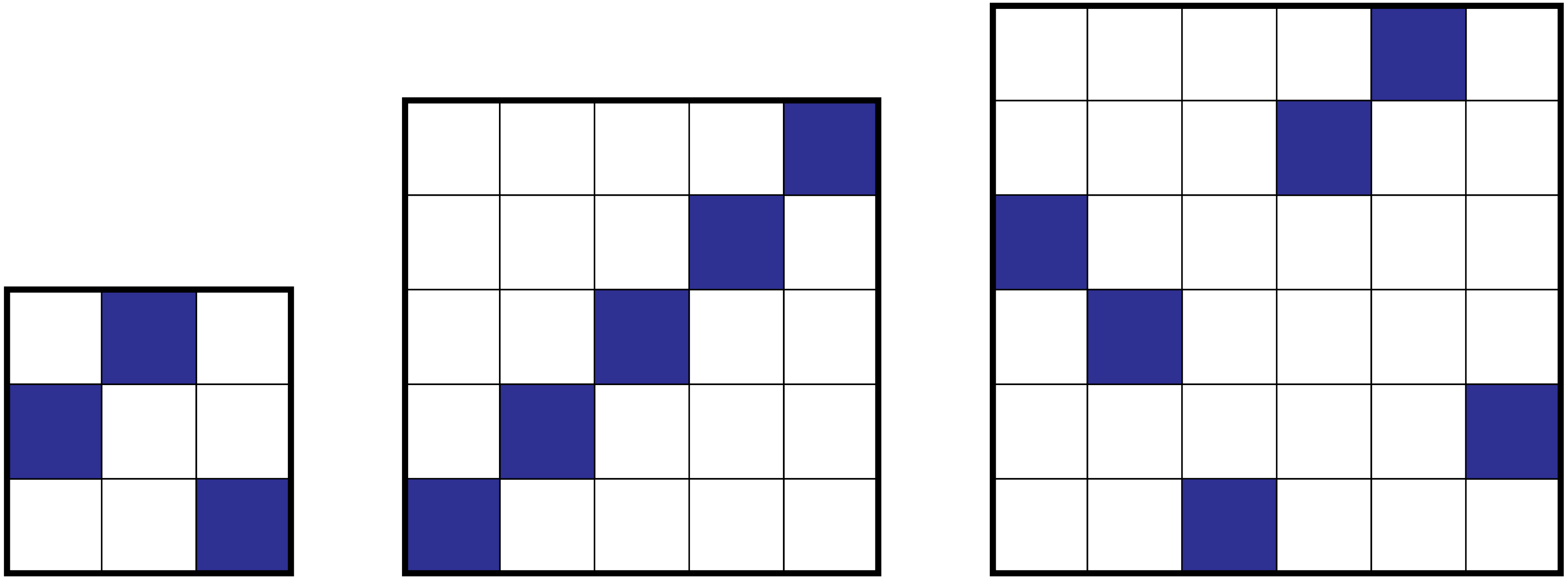


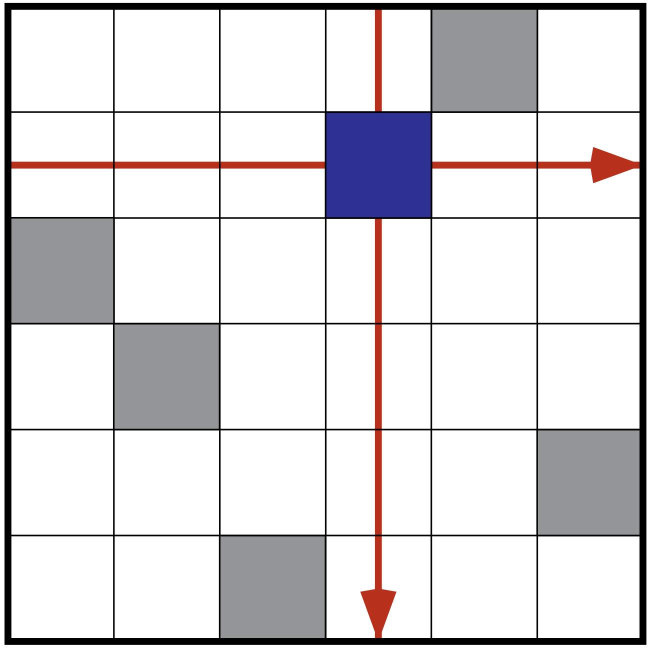
tile $\sim$ transition

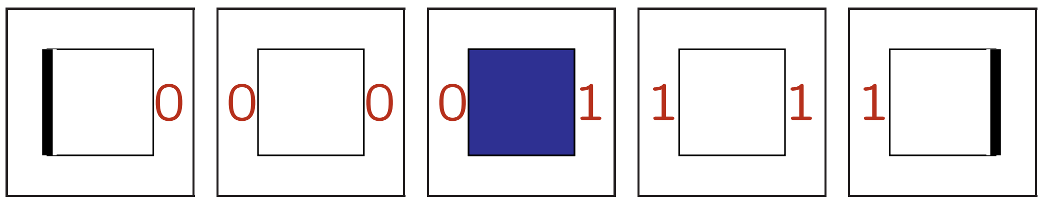


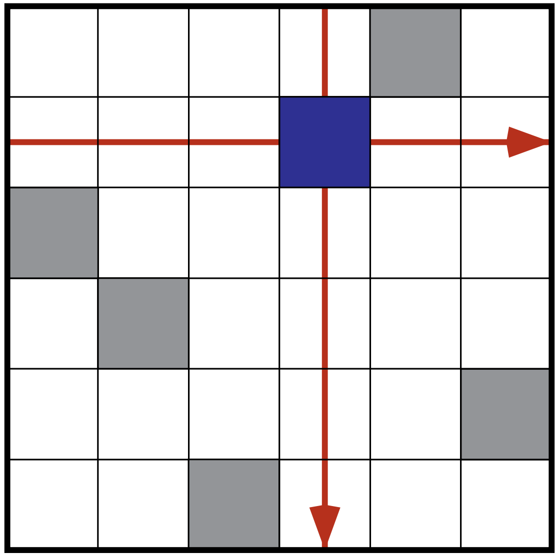


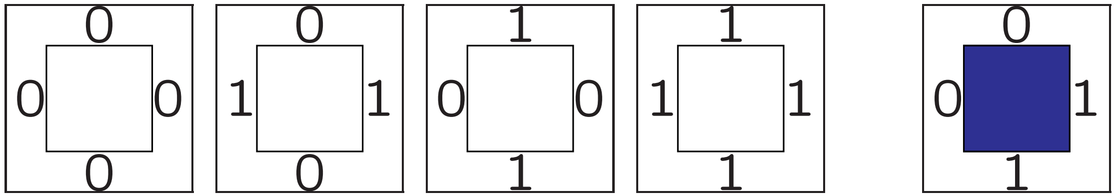
note: hor+vert transitions 'connected'



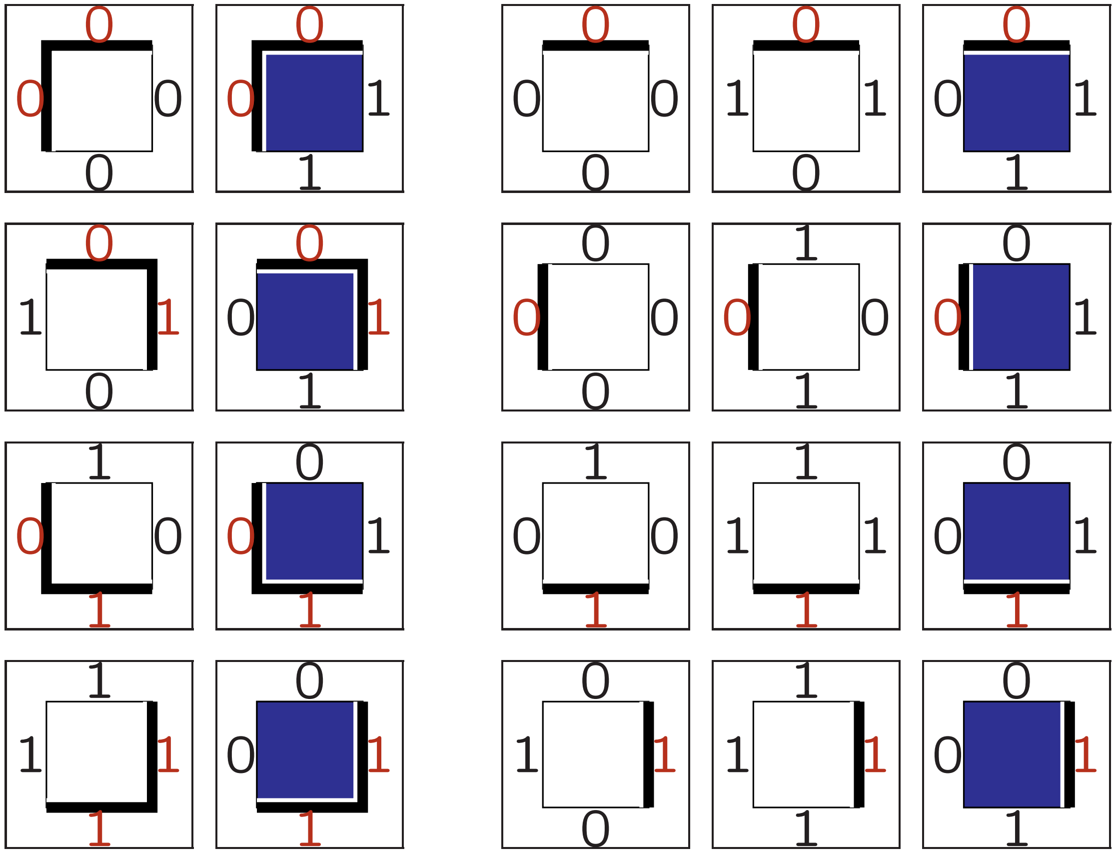



0
0
0
0
1
1
1
1





left/top 0 right/bottom 1

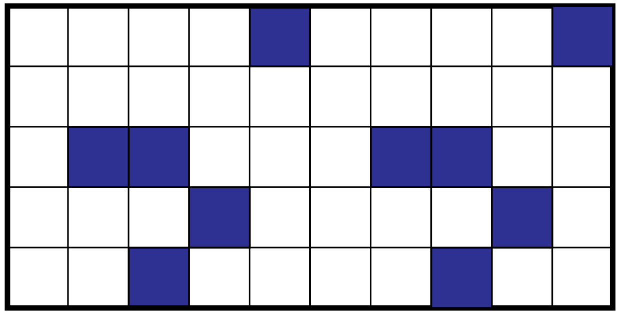


closed under

- renaming
- concatenation, iteration (hor & vert)
- union, intersection
- rotation

- *not* closed under complement

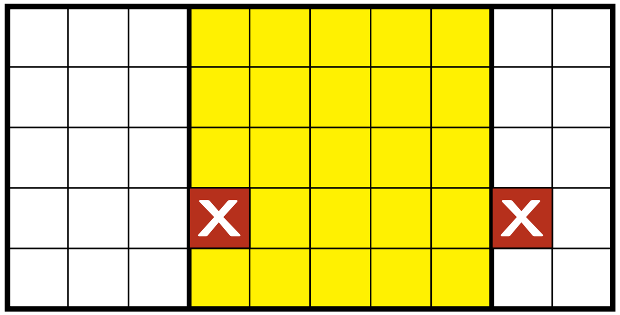
- *not* closed under complement



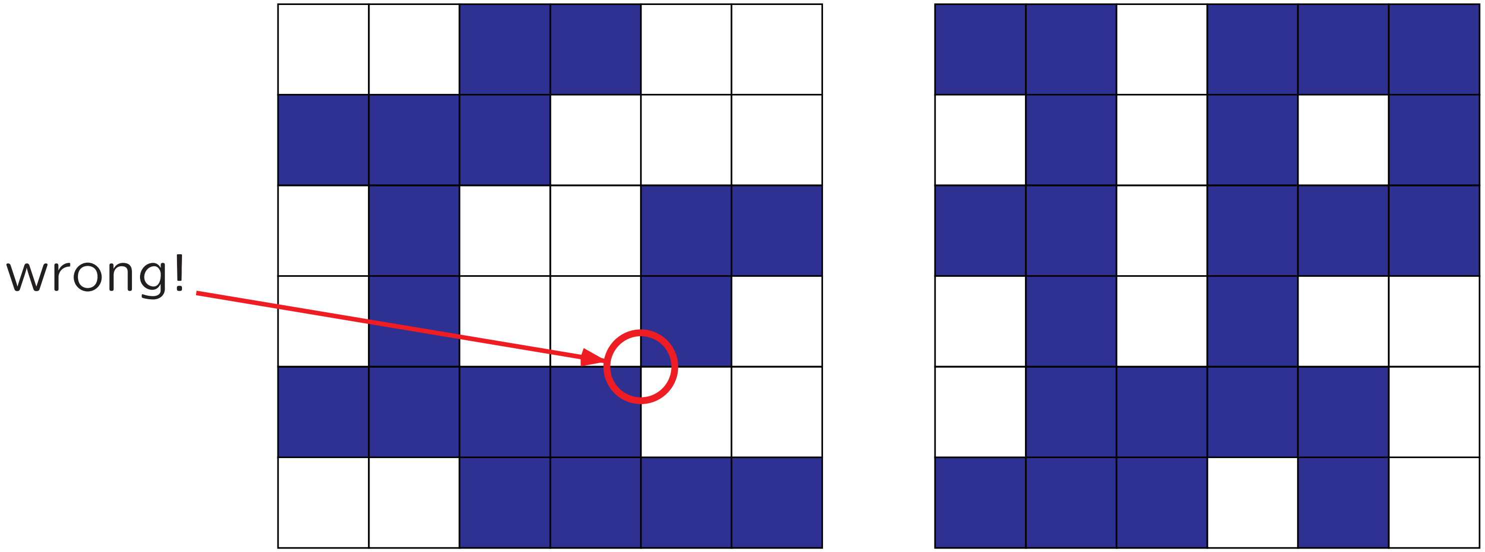
$\{ww \mid w \in \{a,b\}^*\}$
not context-free . . .

c tile colours, e edge colours
choose $c^{n^2} > e^n$, cut & glue

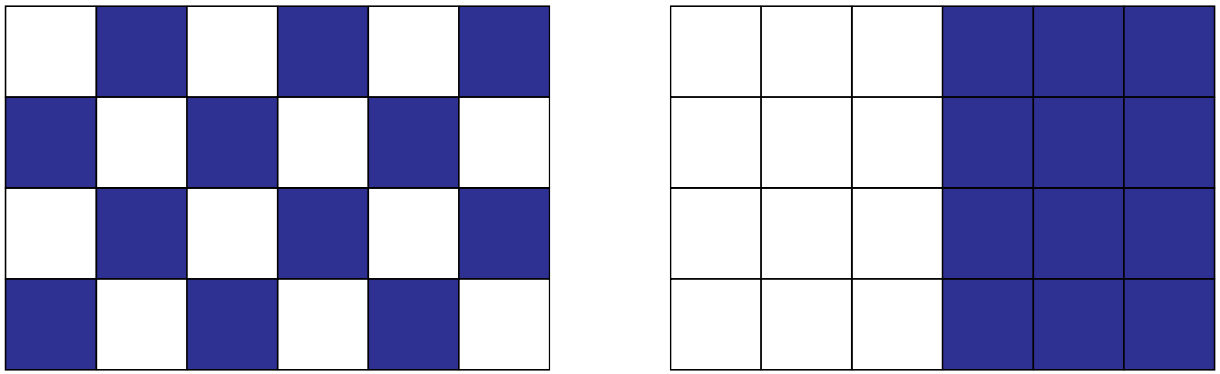
but complement is



- blue tiles are connected (*difficult*)

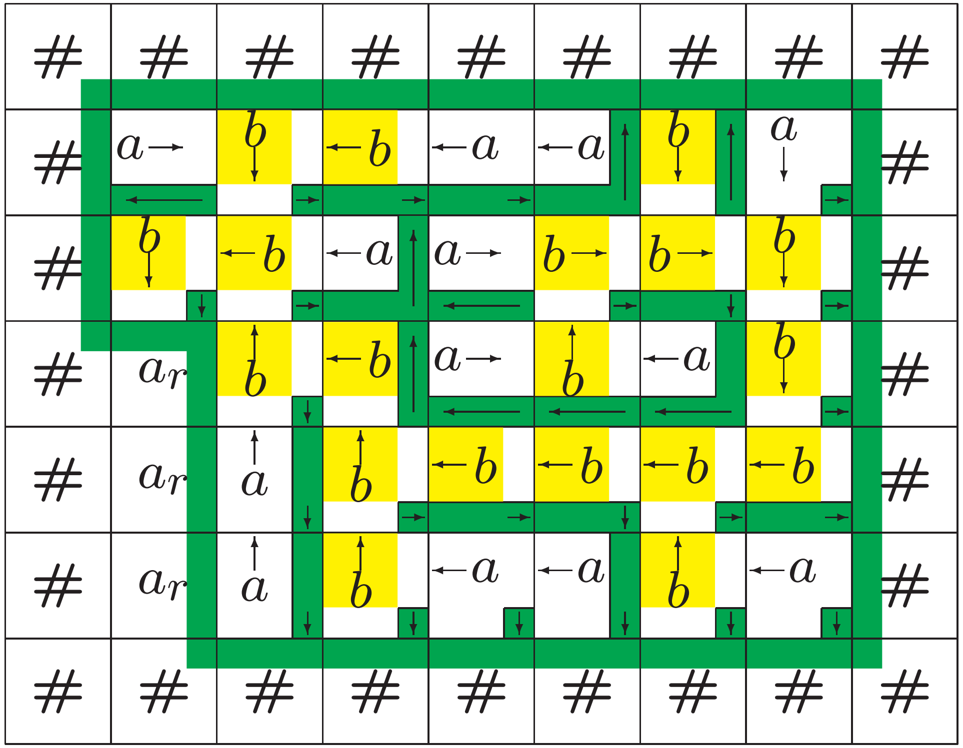


- equal numbers (*very difficult*)



	a	b	b	a	a	b	a	
	b	b	a	a	b	b	b	
	a	b	b	a	b	a	b	
	a	a	b	b	b	b	b	
	a	a	b	a	a	b	a	

using Wang tiles?



Klaus Reinhardt

On Some Recognizable Picture-Languages (1998) logic MSO $\Leftrightarrow$ finite state automata

sets (of positions)

even number of a's

$$
\begin{array}{lll}
(\exists X)[& & (\forall k)(k \in X \rightarrow a(k)) \\
 & \wedge & (\forall i)(\text{first}_a(i) \rightarrow i \in X) \\
 & \wedge & (\forall i)(\forall j)(\text{next}_a(i,j) \rightarrow (i \in X \leftrightarrow j \notin X)) \\
 & \wedge & (\forall i)(\text{last}_a(i) \rightarrow i \notin X) \\
]
\end{array}
$$

where $\text{first}_a(i)$ means

$$a(i) \wedge \neg(\exists j)(a(j) \wedge j < i)$$

etc.

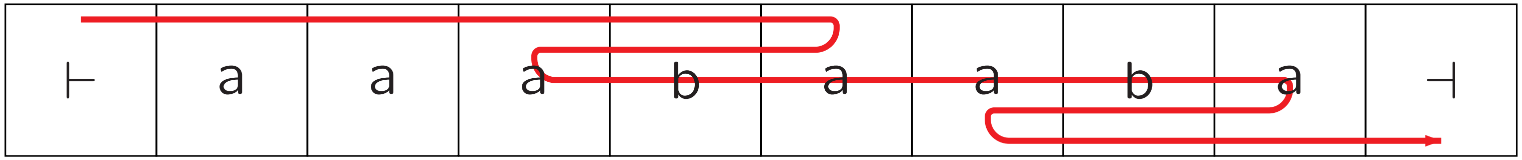
b (a) a b (a) b b a b (a) a b

Prop. EMSO = WANG

■ four-way automata

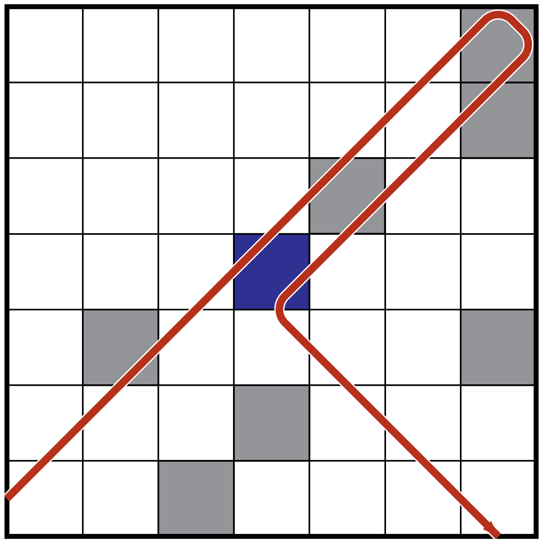
Prop. two-way fsa are equivalent to one-way aut.
(deterministic and non-deterministic)

Rabin&Scott, Shepherdson (1959)

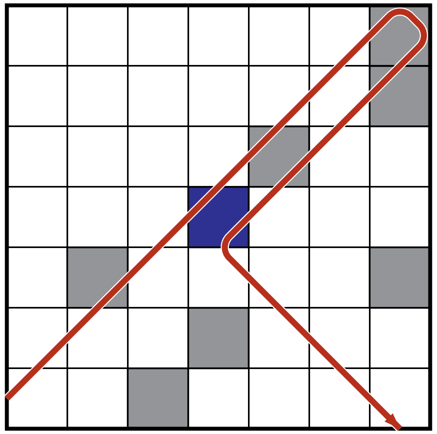


Blum, Hewitt: Automata on a 2-dimensional tape (1967) *can a 'robot' recognize a tiling?*

state $\times$ colour $\longrightarrow$ state $\times$ direction



non-deterministic – guess middle



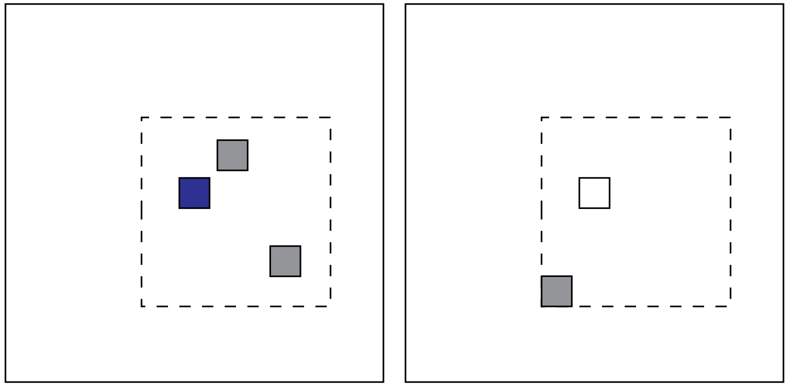
only *non*-deterministically

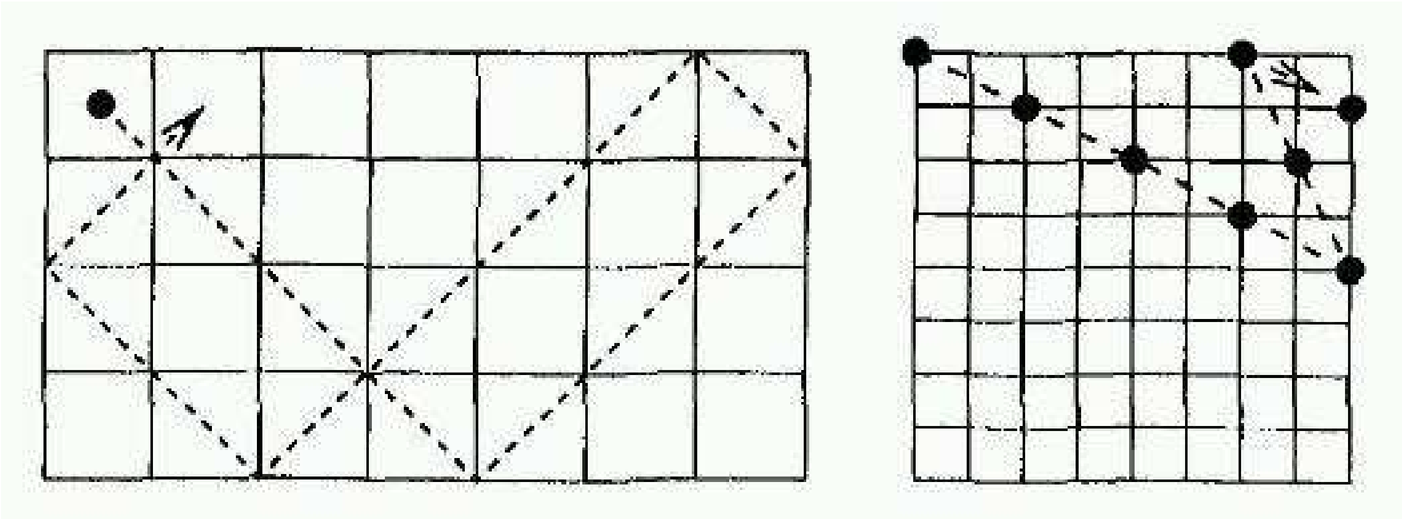
s states, c colours

in-state, out-state (elsewhere)

choose $(4m \cdot s)^{4m \cdot s} < 2^{m^2}$

two equivalent squares



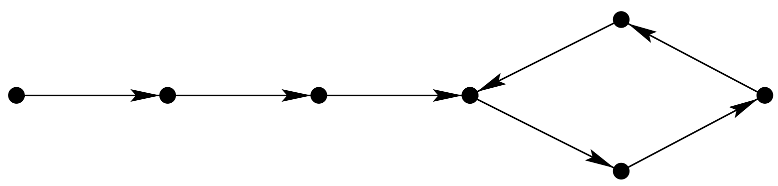


Lindgren, Moore, and Nordahl

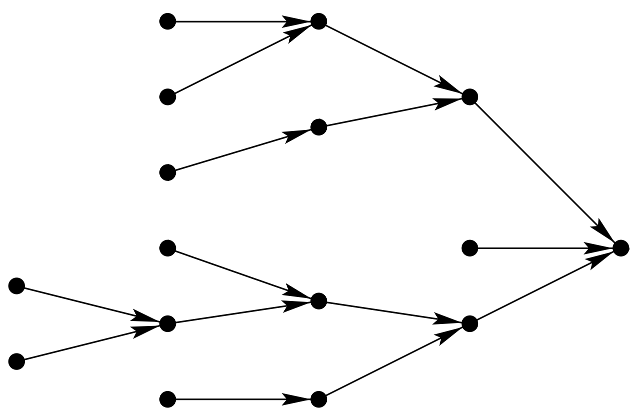
"By bouncing like a billiard ball or making knights' moves, and ending one cell from the corner, a DFA can check that the two sides of a rectangle are mutually prime, or that the side of a square is a power of 2."

"bouncing like a billiard ball . . . can check that . . ."

⇒ avoid loops!



simulate search space backwards: a tree!



this can be done by a 4DFA !

Sipser, Halting space-bounded computations (1980) deterministic closed under

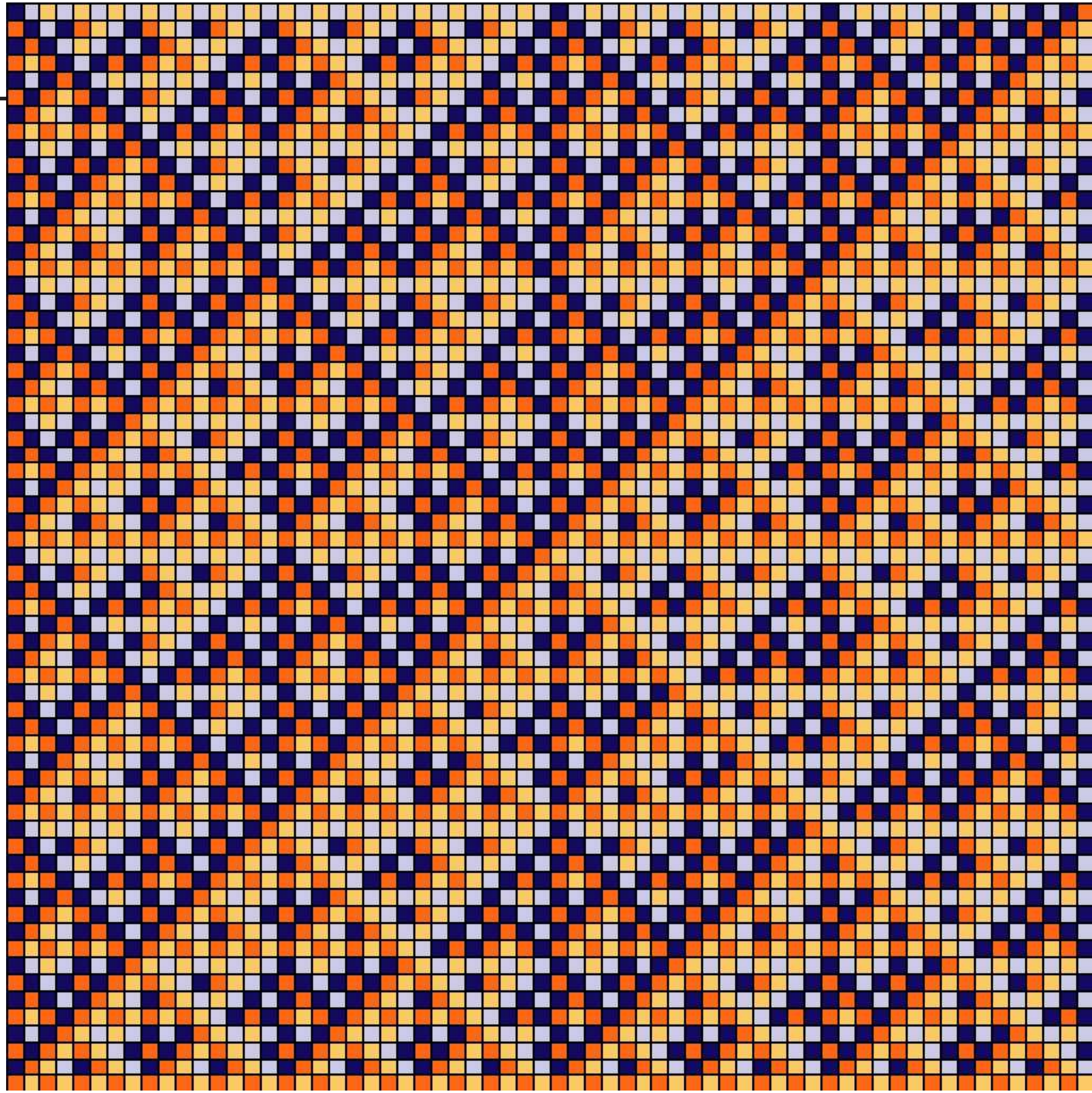
- boolean union, intersection, complement
- rotation
- *not* closed under concatenation, iteration
  'regular'

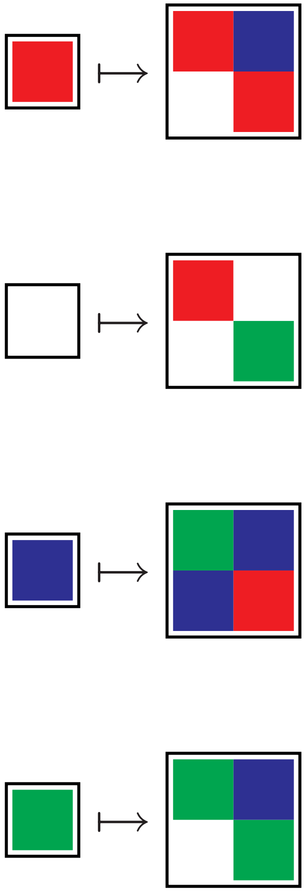
non-deterministic closed under

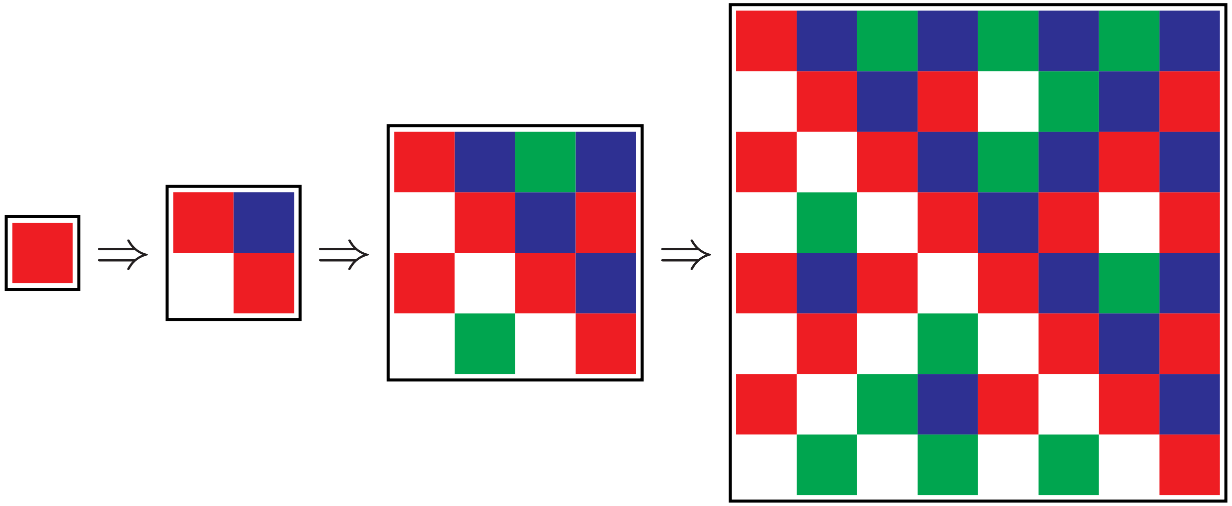
- union, intersection,
- rotation
- *not* closed under concatenation, iteration, complement

4DFA $\subset$ 4NFA $\subset$ WANG

■ Iterated substitutions







square chair tiling

grammar-like

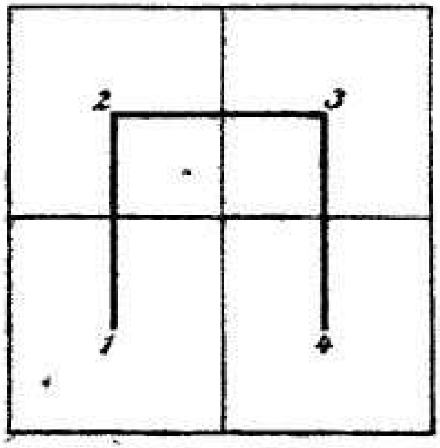


Fig. 1.

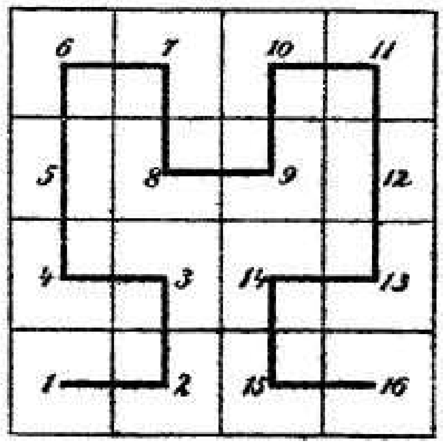


Fig. 2.

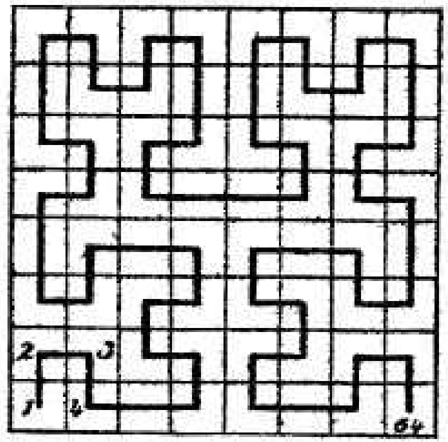
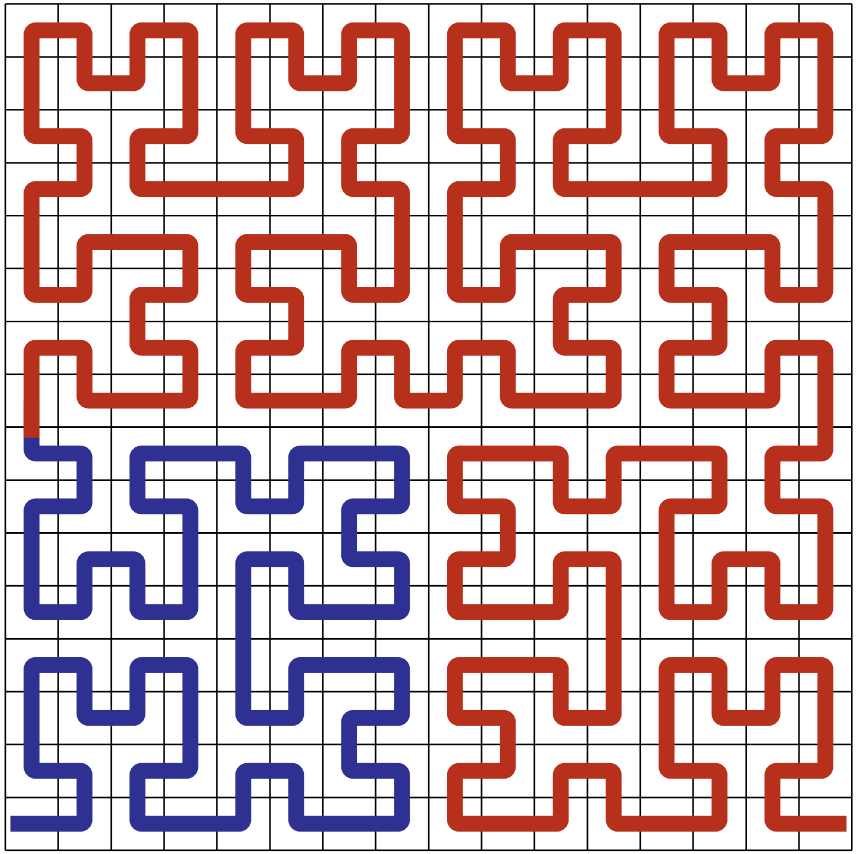


Fig. 3.

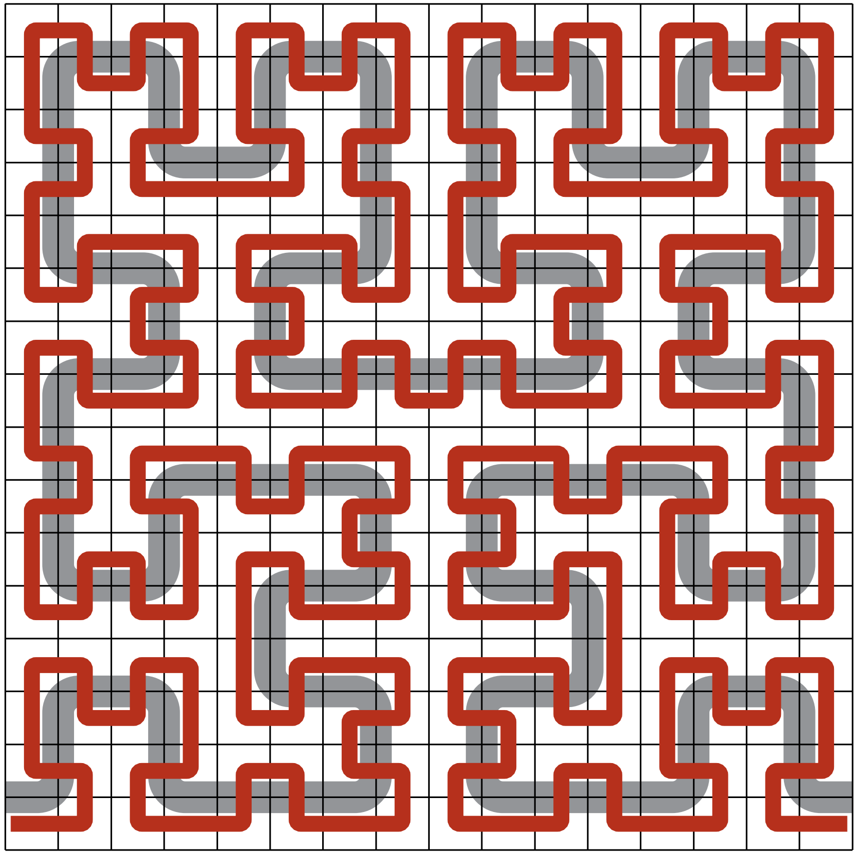
Ueber die stetige Abbildung einer Linie auf ein Flächenstück (1891)

David Hilbert in Königsberg i. Pr.



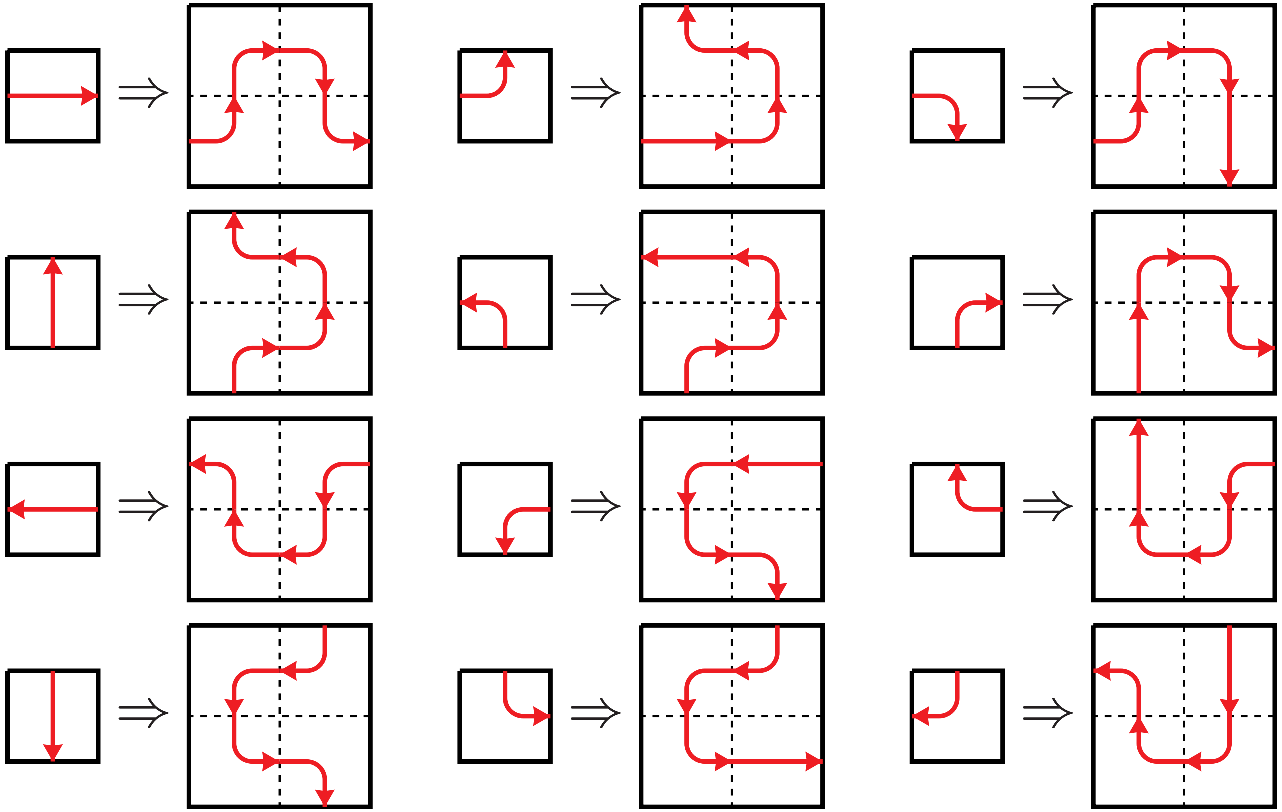
four copies

top



refinement

bottom

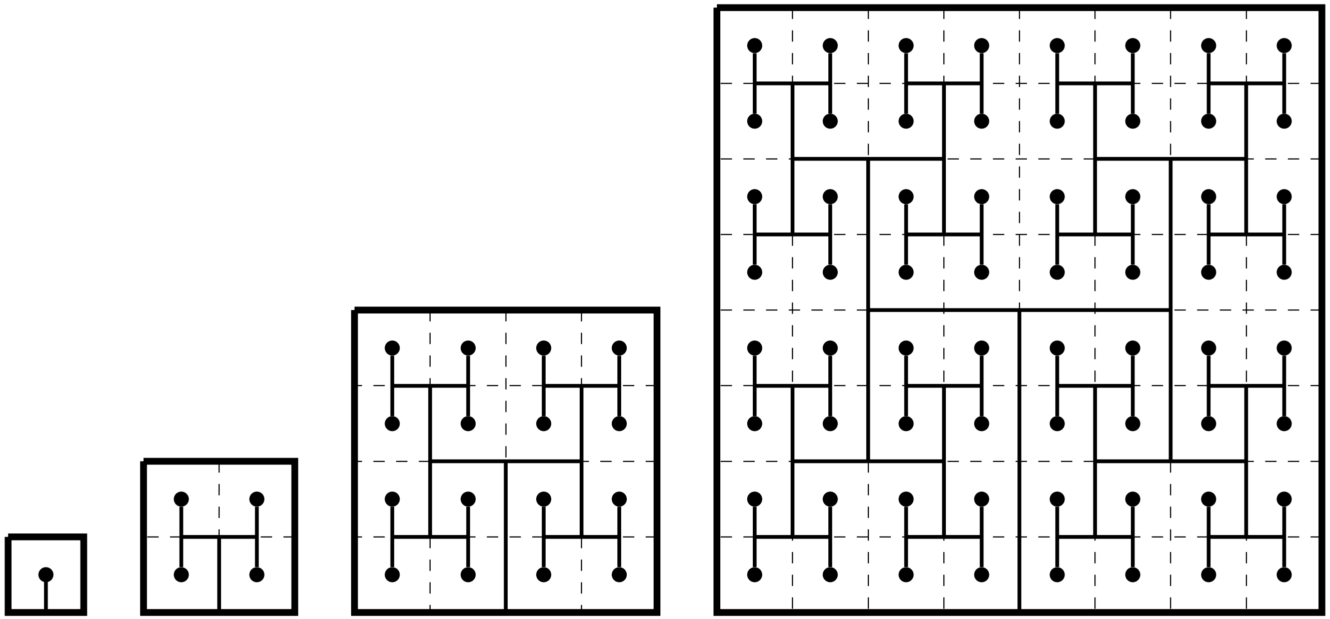


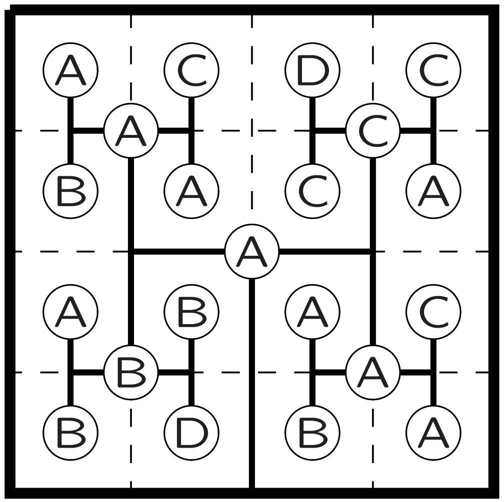
twelve tiles

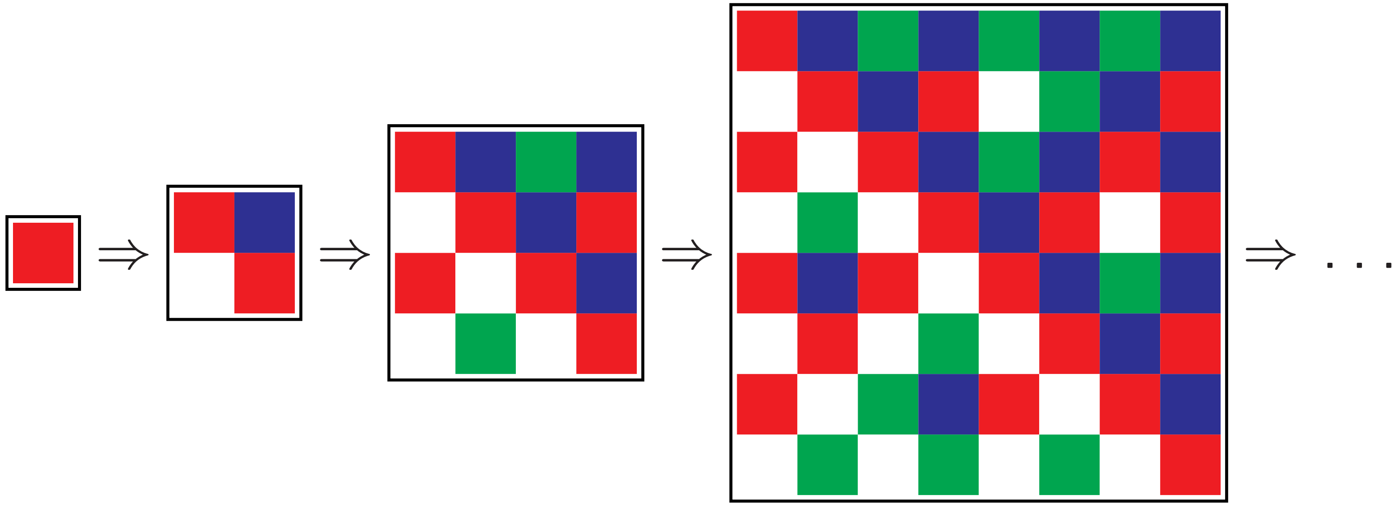
Proposition. Every picture language defined by a substitution can be Wang tiled

Lindgren, Moore, Nordahl: Complexity of two-dimensional patterns (1998)

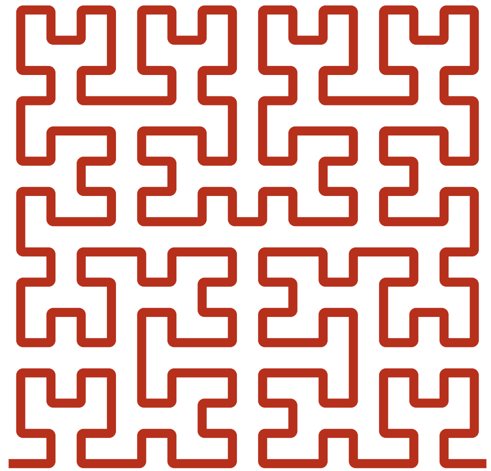
Mozes: Tilings, Substitution Systems and Dynamical Systems Generated by Them (1989).



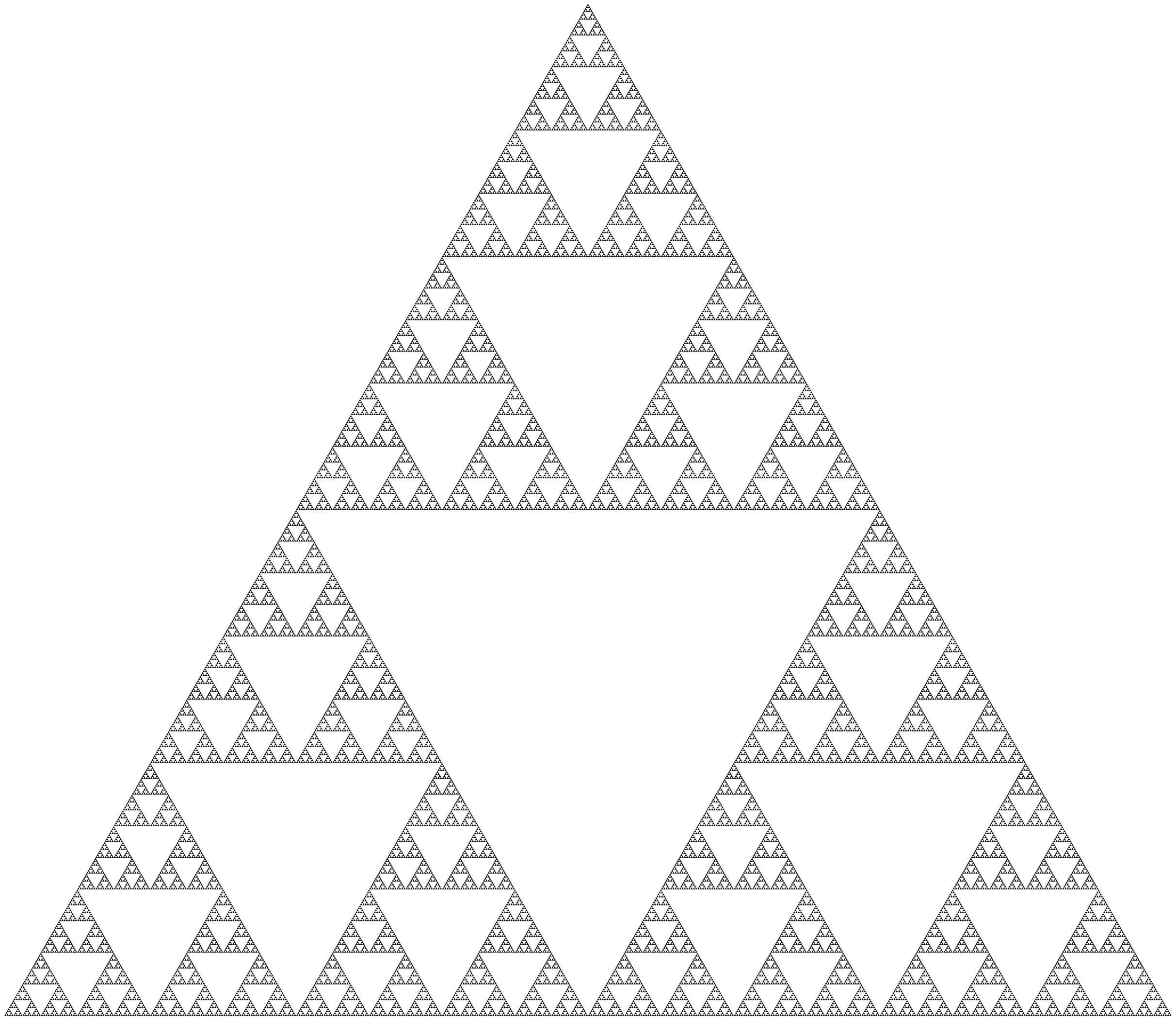

A C D C
A C
B A C A
A
A B A C
B A
B D B A



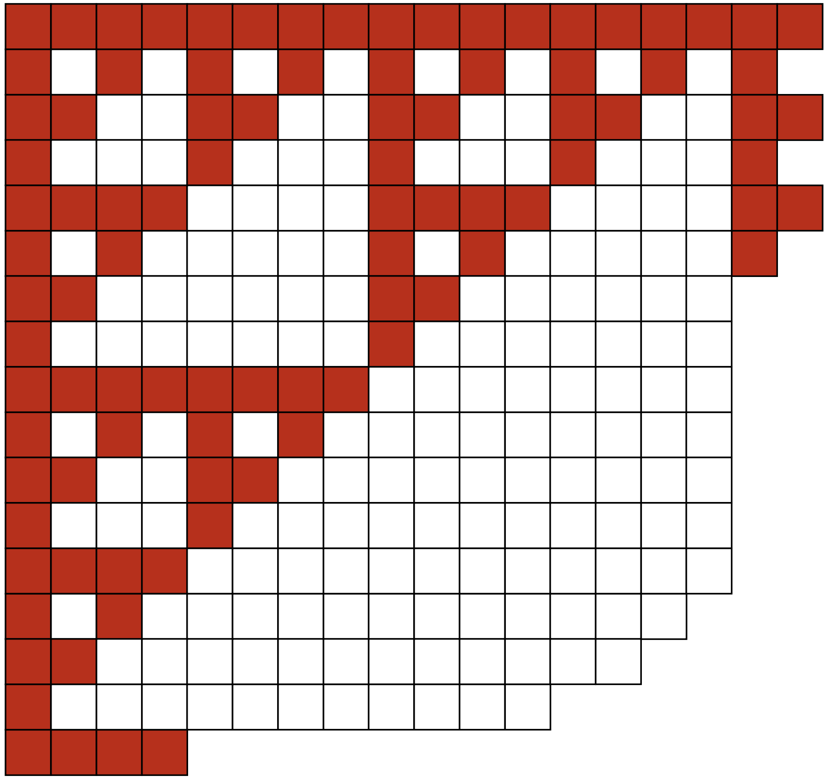
iterated substitutions form patterns that can be tiled

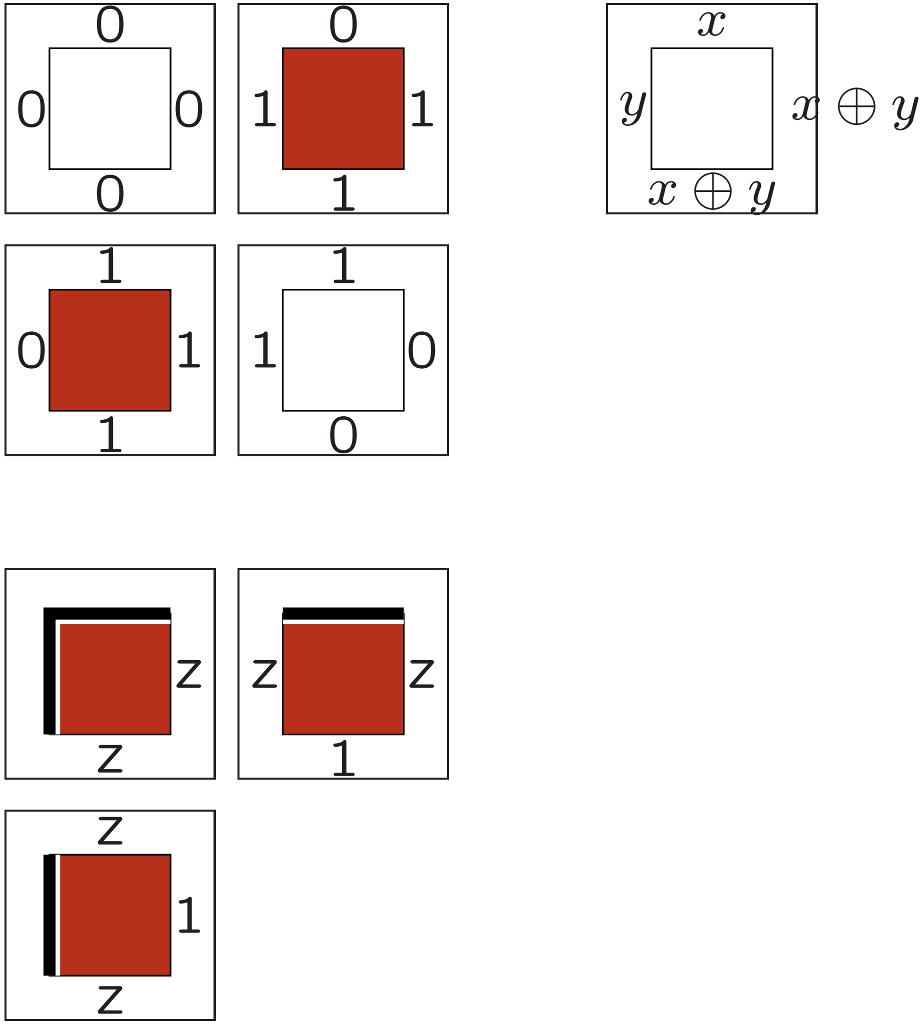


■ Self-Assembly

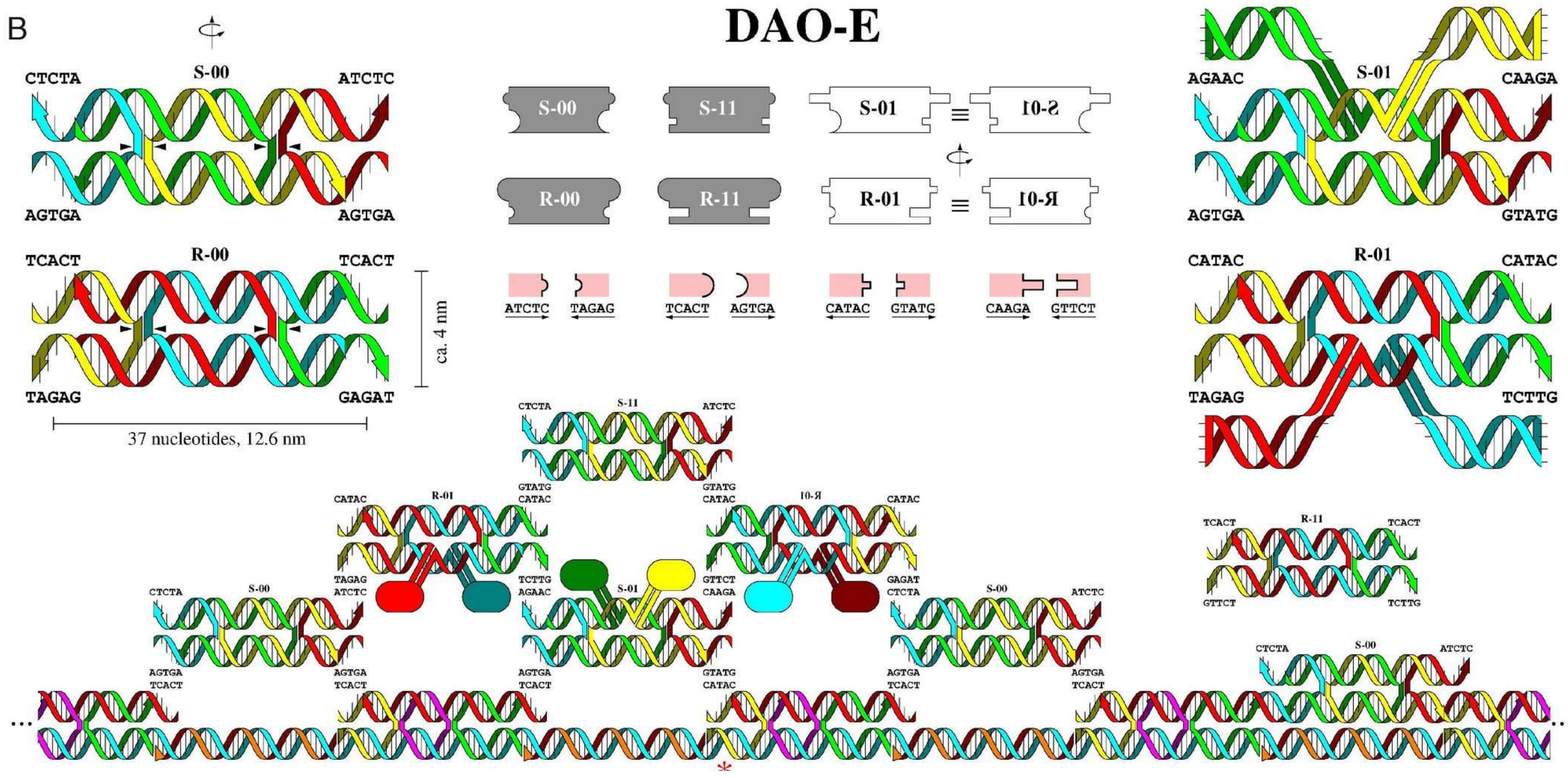


Qef's Website
wikipedia



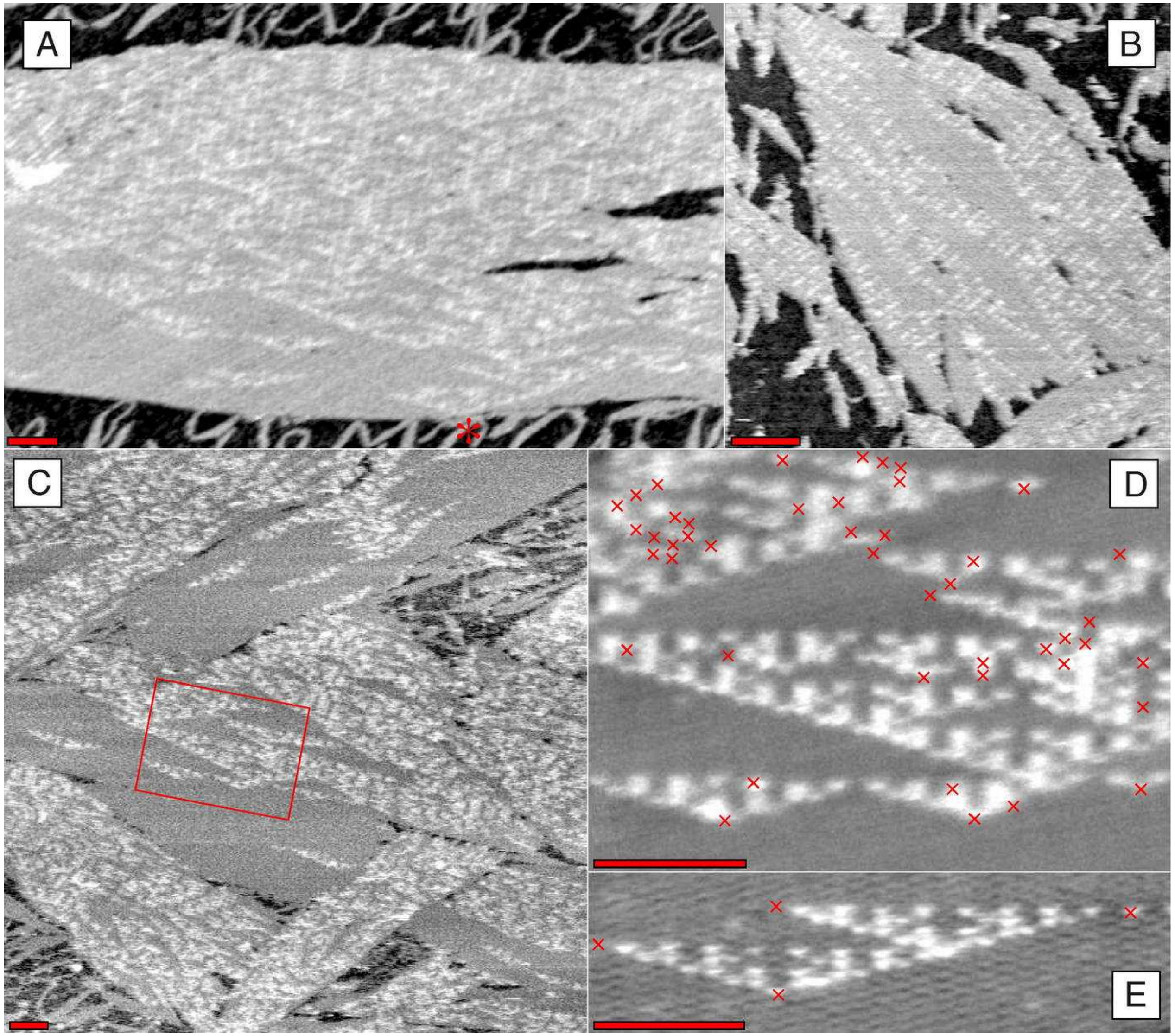


x	y	$x \oplus y$
0	0	0
0	1	1
1	0	1
1	1	0

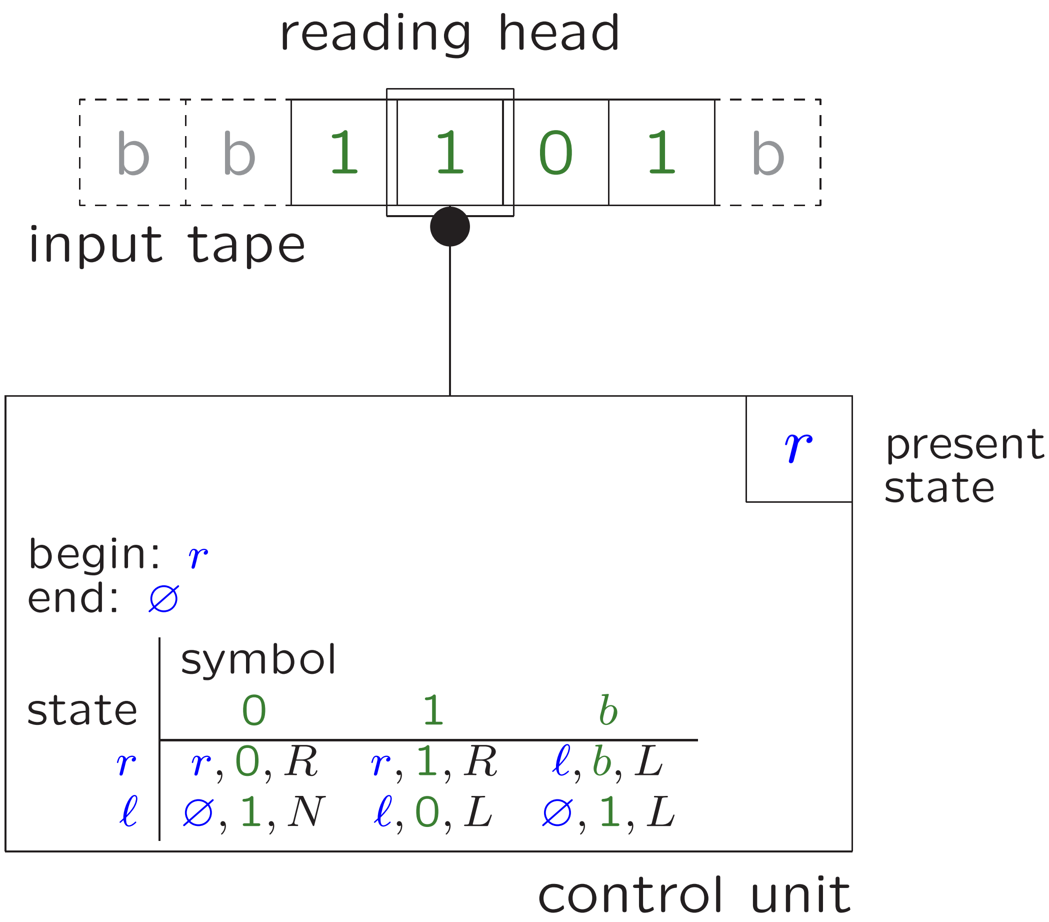


Algorithmic Self-Assembly of DNA Sierpinski Triangles (2004)

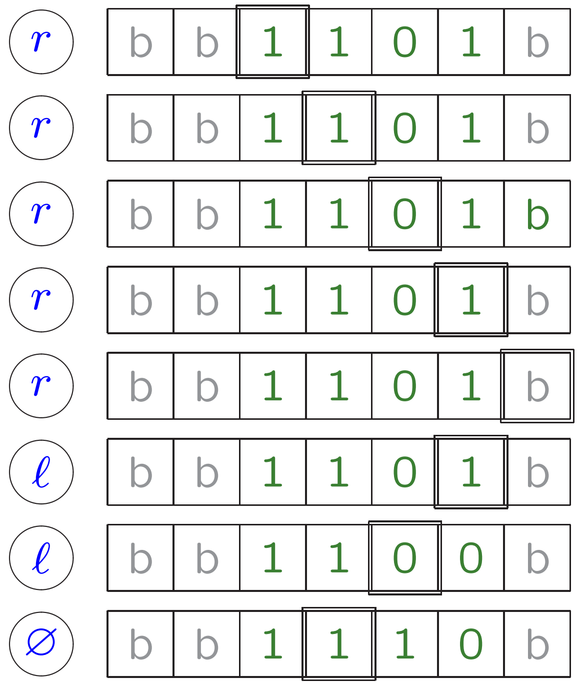
Rothemund, Papadakis, Winfree; PLoS Biology


A
B
C
D
E

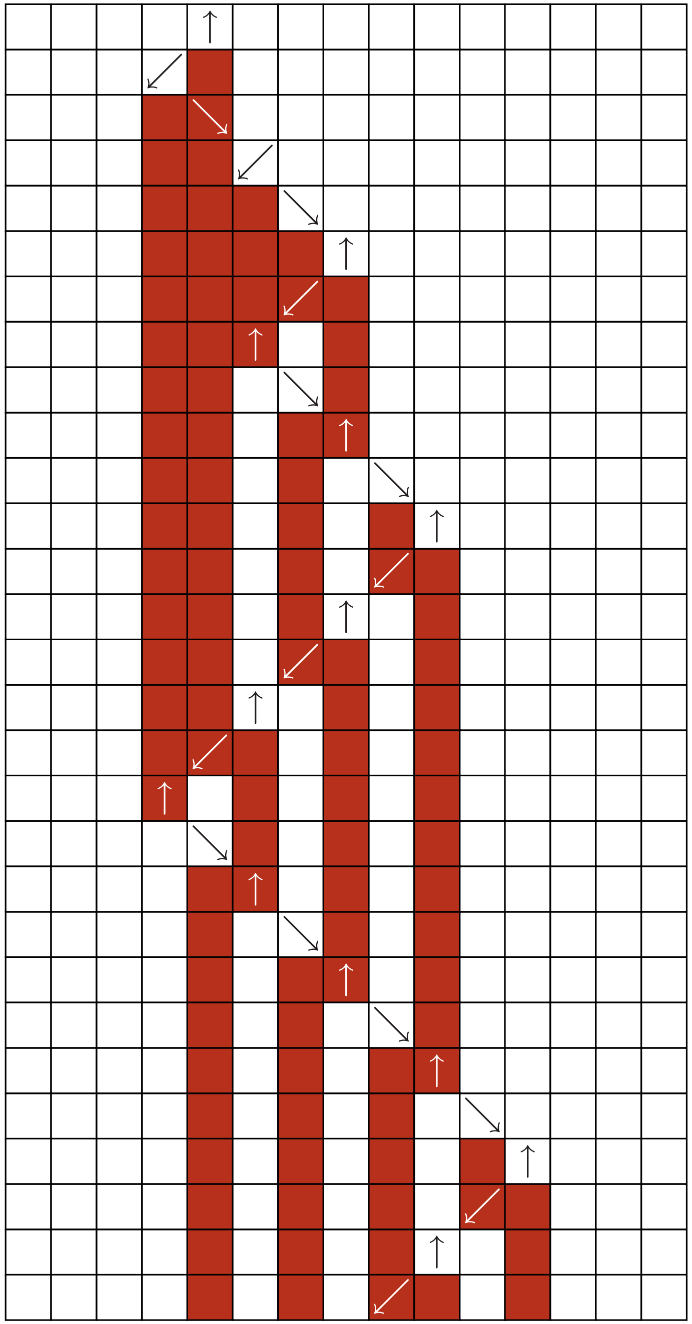
■ no algorithm



state \ symbol	0	1	b
r	$r, 0, R$	$r, 1, R$	ℓ, b, L
ℓ	$\varnothing, 1, N$	$\ell, 0, L$	$\varnothing, 1, L$

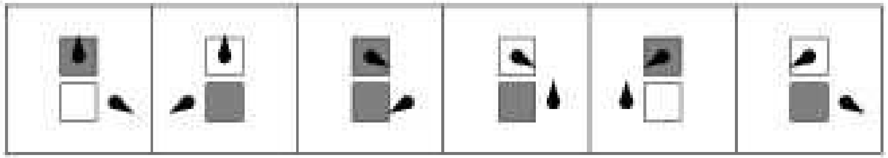


Turing, A. M. *On Computable Numbers, with an Application to the Entscheidungsproblem.* Proc. London Math. Soc. Ser. 2 42, 230-265, 1937.



3 states,

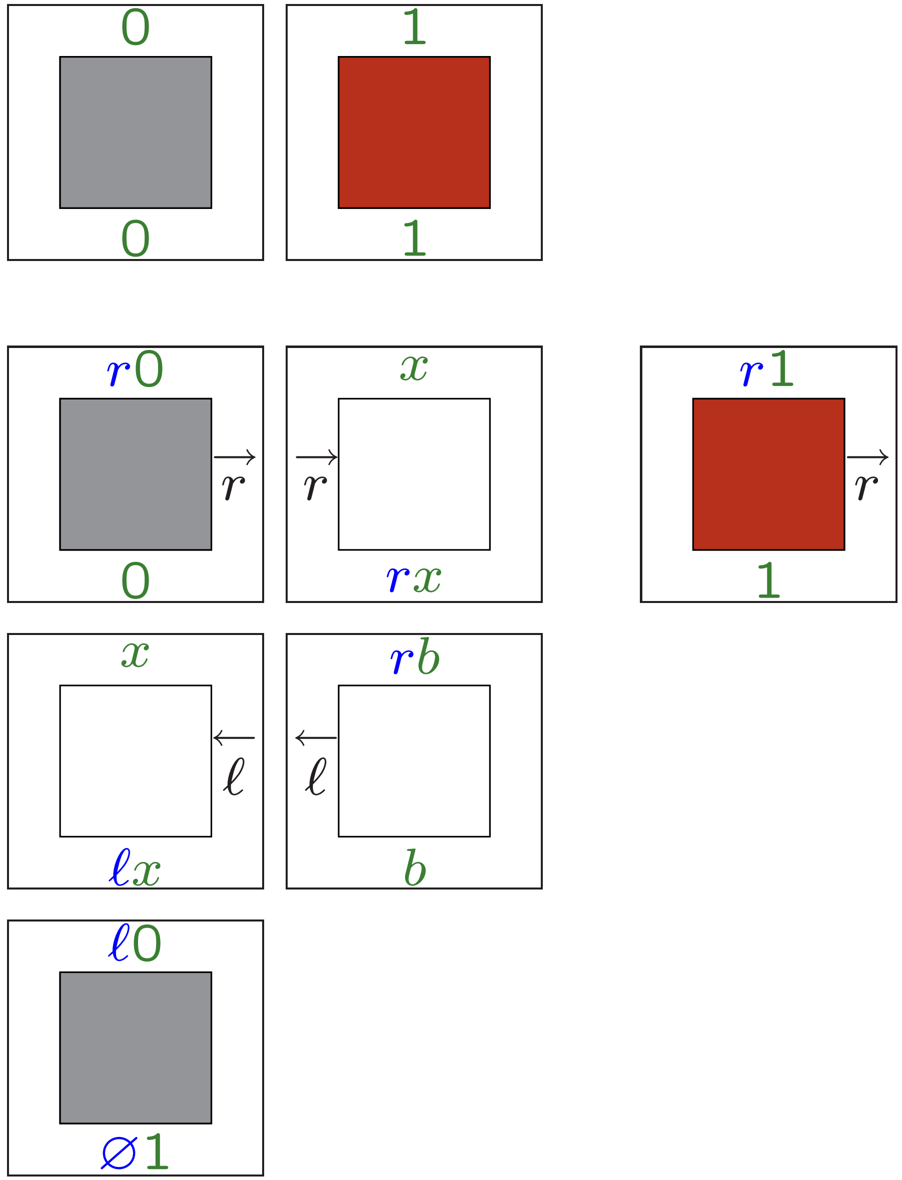
2 symbols (colours)



	0	1
↑	↙, 1, L	↘, 0, R
↙	↘, 1, R	↑, 0, L
↘	↑, 1, R	↙, 1, R

r0	1	0	1	1	B
0	r1	0	1	1	B
0	1	r0	1	1	B
0	1	0	r1	1	B
0	1	0	1	r1	B
0	1	0	1	1	rB
0	1	0	1	ℓ1	B
0	1	0	ℓ1	0	B
0	1	ℓ0	0	0	B
0	1	1	0	0	B

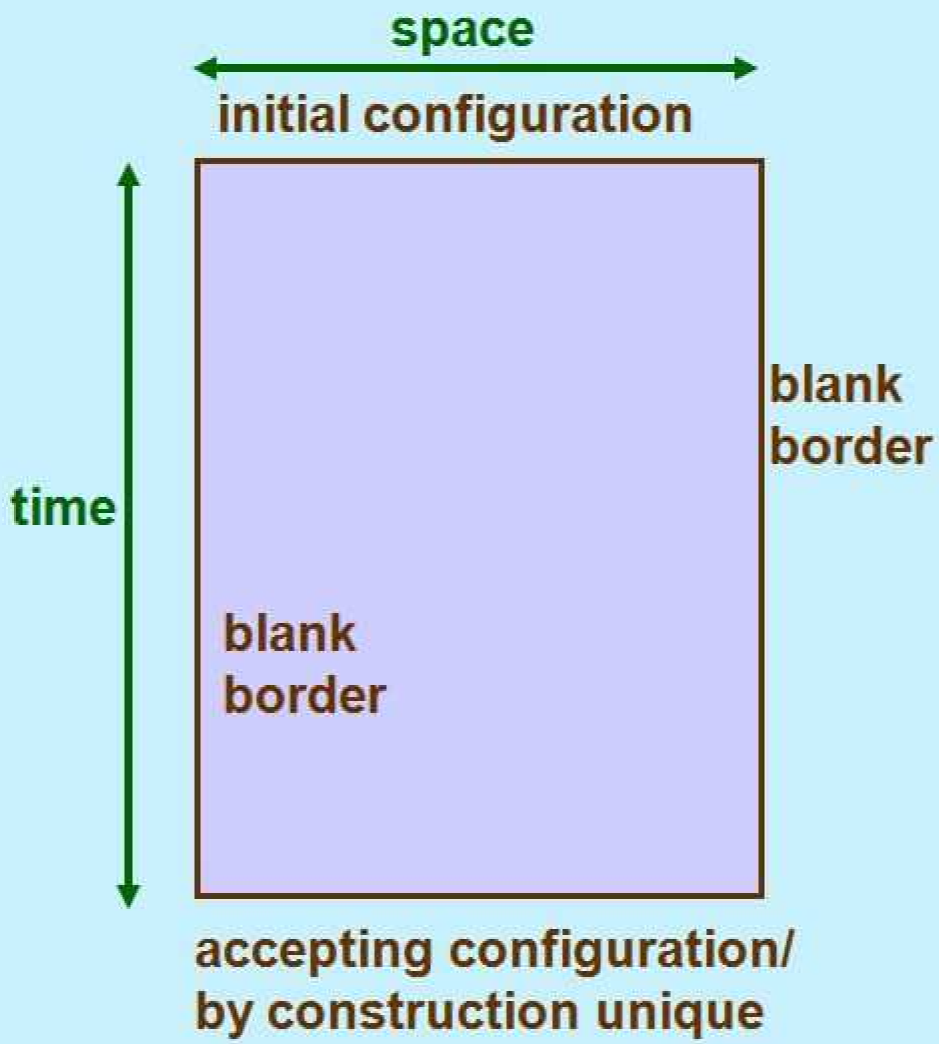
	0	1	b
r	$r, 0, R$	$r, 1, R$	ℓ, b, L
ℓ	$\varnothing, 1, N$	$\ell, 0, L$	$\varnothing, 1, L$



Tiling reductions

Program : Tile Types
Input: Boundary condition

Space: Width region
Time: Height region

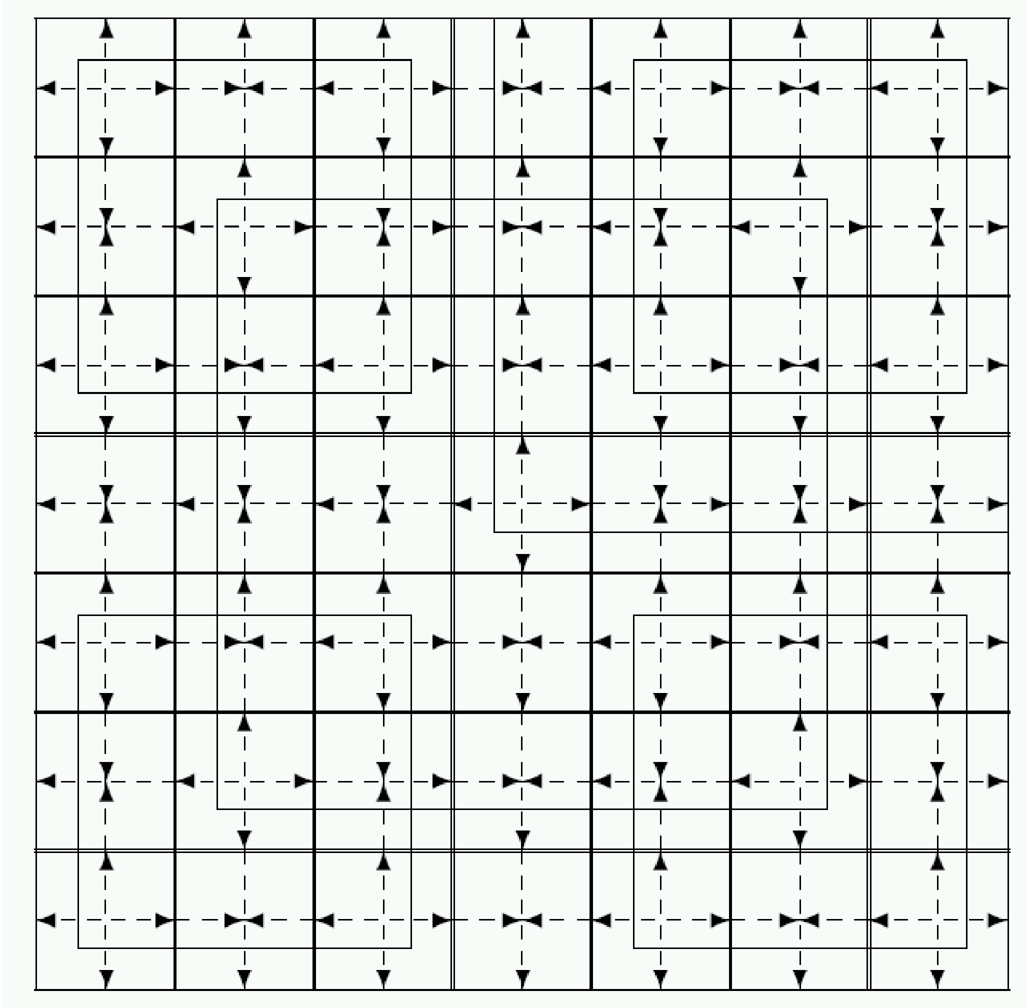


Square Tiling: A given square with boundary condition: Complete for NP.

Corridor Tiling: A rectangle with boundary conditions on entrance and exit (length is undetermined): Complete for PSPACE .

Origin Constrained: The entire plane with a given Tile at the Origin. Complete for co-RE
hence Undecidable

General: The entire plane without constraints. Still Complete for co-RE (Wang/Berger's Theorem). Hard to Prove!



Robinson

Bedankt!

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