Data Structures

September 28

Hierachical structures: Trees

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Objectives

Discuss the following topics:

- Trees, Binary Trees, and Binary Search Trees
- Implementing Binary Trees
- Tree Traversal
- Searching a Binary Search Tree
- Insertion
- Deletion

Objectives (continued)

Discuss the following topics:

- Heaps
- Balancing a Tree
- Self-Adjusting Trees

Trees, Binary Trees, and Binary Search Trees

- A **tree** is a data type that consists of **nodes** and **arcs**
- These trees are depicted upside down with the root at the top and the **leaves** (**terminal nodes**) at the bottom
- The **root** is a node that has no parent; it can have only child nodes
- Leaves have no children (their children are null)

Trees, Binary Trees, and Binary Search Trees (continued)

- Each node has to be reachable from the root through a unique sequence of arcs, called a **path**
- The number of arcs in a path is called the **length** of the path
- The **level** of a node is the length of the path from the root to the node plus 1, which is the number of nodes in the path
- The **height** of a nonempty tree is the maximum level of a node in the tree

Trees, Binary Trees, and Binary Search Trees (continued)

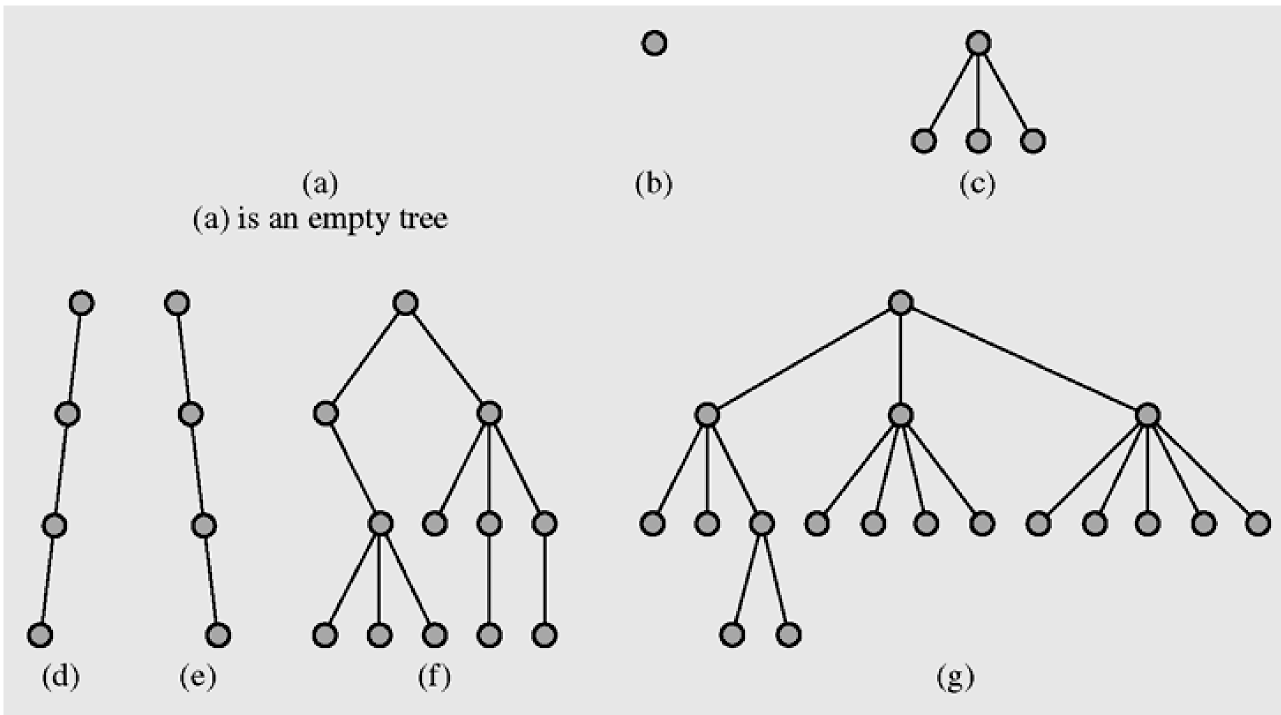


Figure 6-1 Examples of trees

Trees, Binary Trees, and Binary Search Trees (continued)

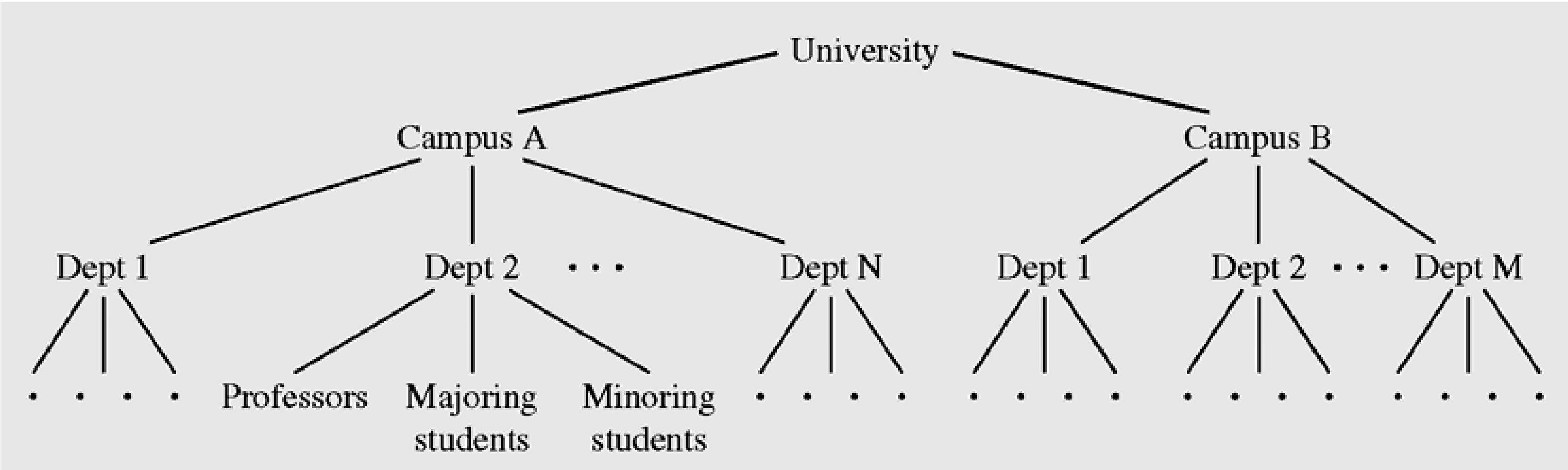


Figure 6-2 Hierarchical structure of a university shown as a tree

Trees (adt)

Trees: abstract/mathematical

important, great number of varieties

- terminology

    (knoop, wortel, vader, kind)
    node/vertex, root, father/parent, child
    (non) directed
    (non) orderly
    binary trees (left ≠ right)
    full (sometimes called decision trees, see Drozdek), complete (all levels are filled, except the last one)

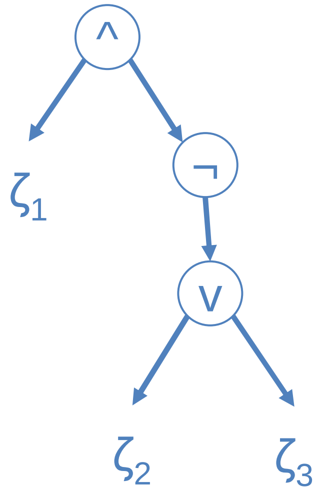
- categorization

structure

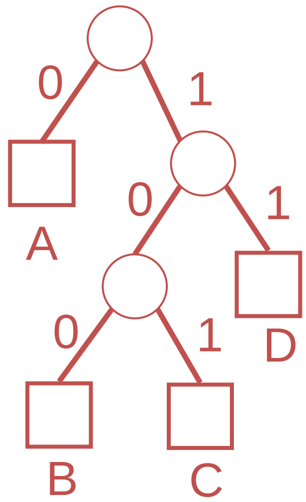
    number of children (binary, B-boom)
    height of subtrees (AVL-, B-trees)
    compleet (heap)

Location of keys

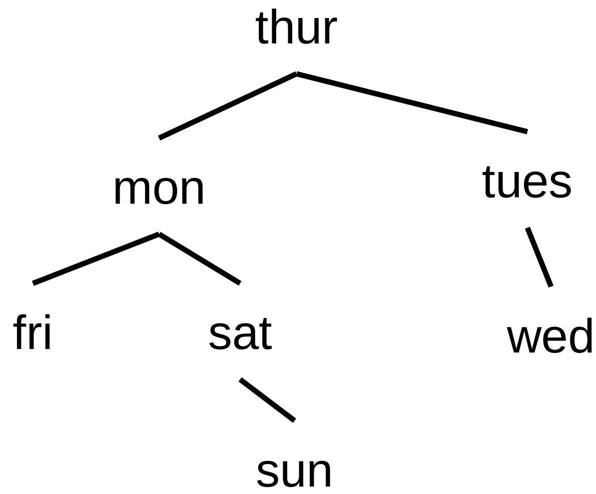
    search tree, heap



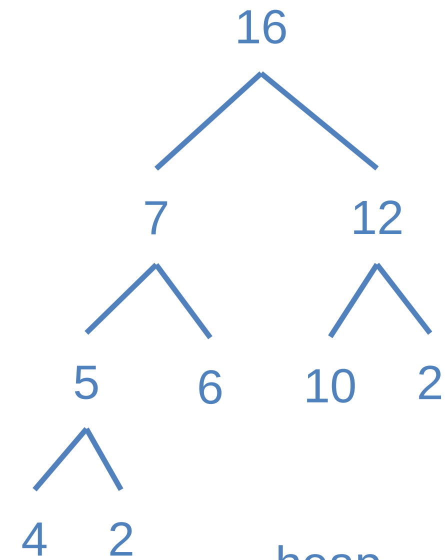
expression



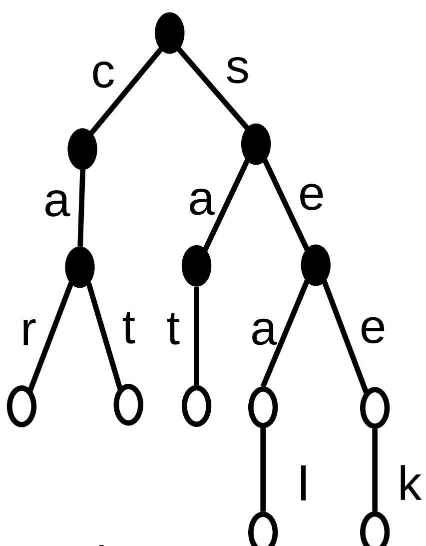
code



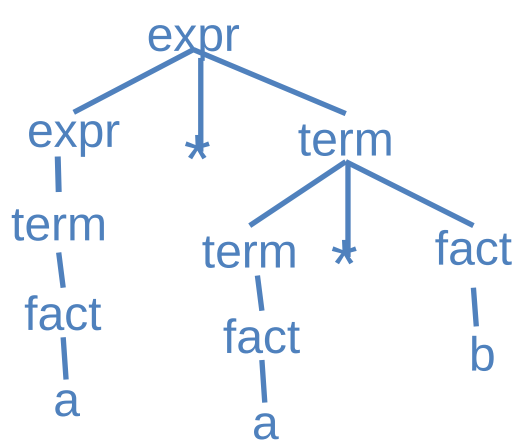
bst



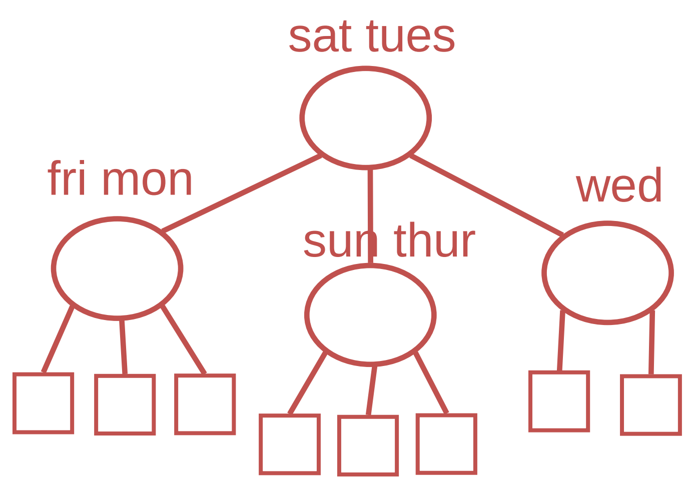
heap



trie



syntax



B-tree (2,3 tree)

Recall Definition of Tree

1. An empty structure is a tree
2. If t1, ..., tk are disjoint trees, then the structure whose root has as its children the roots of t1,...,tk is also a tree
3. Only structures generated by rule 1 and 2 are trees

Alternatively: a connected graph which contains no cycles is a tree

Equivalent statements (see ϕ1)

- Let T be graph with n vertices then the following are equivalent:
  - a) T is a tree (= no cycles and connected)
  - b) T contains no cycles, and has n-1 edges
  - c) T is connected, and has n-1 edges
  - d) T is connected, and every edge is a bridge
  - e) Any two vertices are connected by exactly one path
  - f) T contains no cycles, but the addition of any new edge creates exactly one circuit (cycle with no repeated edges).

Trees, Binary Trees, and Binary Search Trees (continued)

- An **orderly tree** is where all elements are stored according to some predetermined criterion of ordering

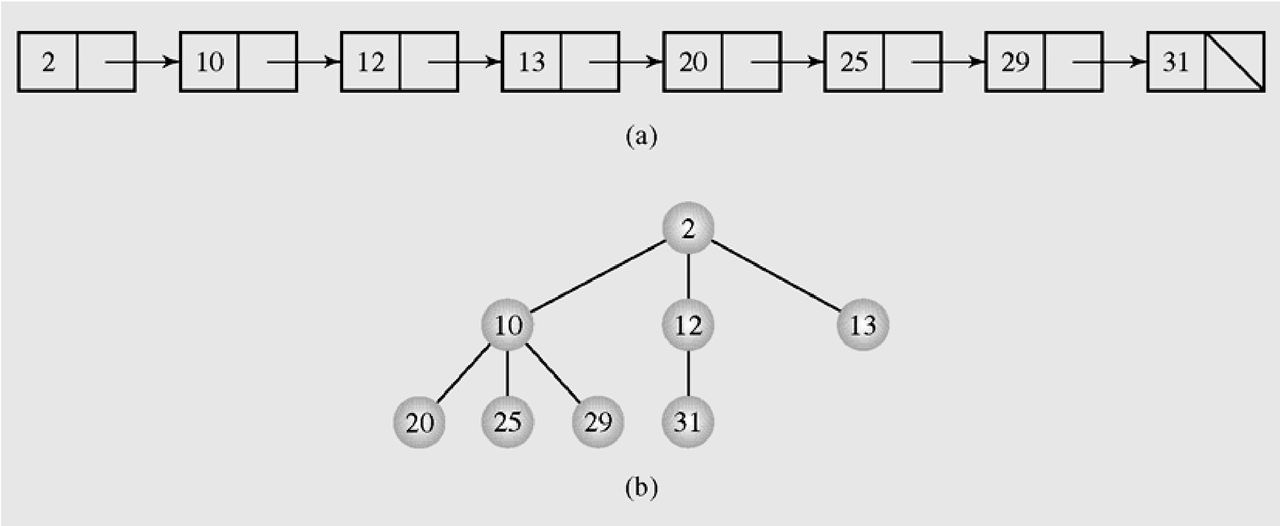


Figure 6-3 Transforming (a) a linked list into (b) a tree

Binary Trees

- A **binary tree** is a tree whose nodes have two children (possibly empty), and each child is designated as either a left child or a right child

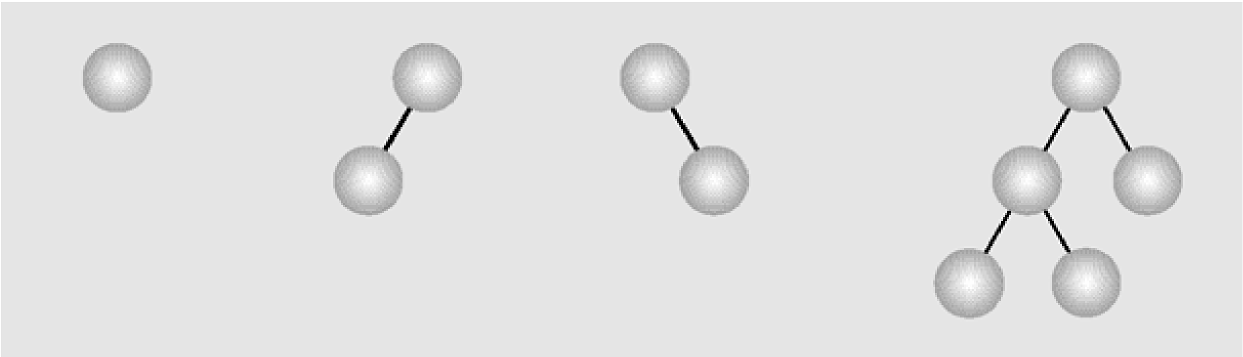
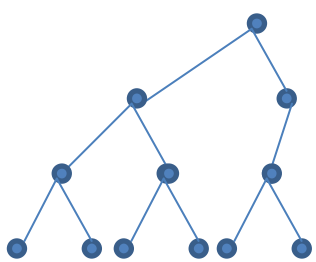


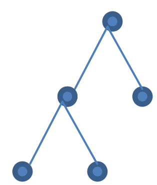
Figure 6-4 Examples of binary trees

Binary Trees

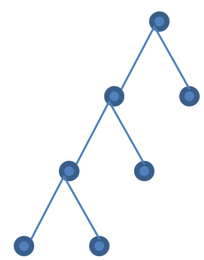
- In a **complete binary tree**, all the nodes at all levels have two children except the last level.
- A **decision tree** is a binary tree in which all nodes have either zero or two nonempty children



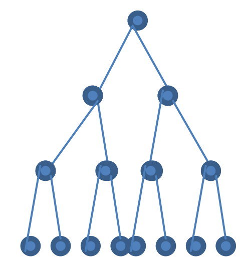
incomplete
Binary tree



Complete
Binary tree
Dutch: compleet)
The more common
def



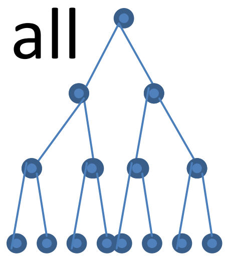
Decision tree
(Dutch: vol)



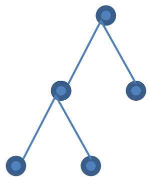
complete
Tree (Drozdek def)

Remark on definition of Complete Binary Tree

- Drozdek Page218 uses the following definition: a complete binary tree is a binary tree of which all non-terminal nodes have both their children, and all leaves are at the same level



- The more common definition is as follows: A complete binary tree is a binary tree in which every level, except possibly the last, is completed filled, and all nodes are as far left as posssible



Binary Trees

- At level i in binary tree at most 2^{i-1} nodes
- For non-empty binary tree whose nonterminal nodes (i.e., a full binary tree) have exactly two nonempty children: #of leaves = 1+#nonterminal nodes
- In a Drozdek-complete tree: # of nodes = $2^{height}-1$; one way to see this is to use the statement #of leaves = 1+#nonterminal nodes; another way is to count how many nodes there are in each level and then sum the geometric progression;

Binary Trees

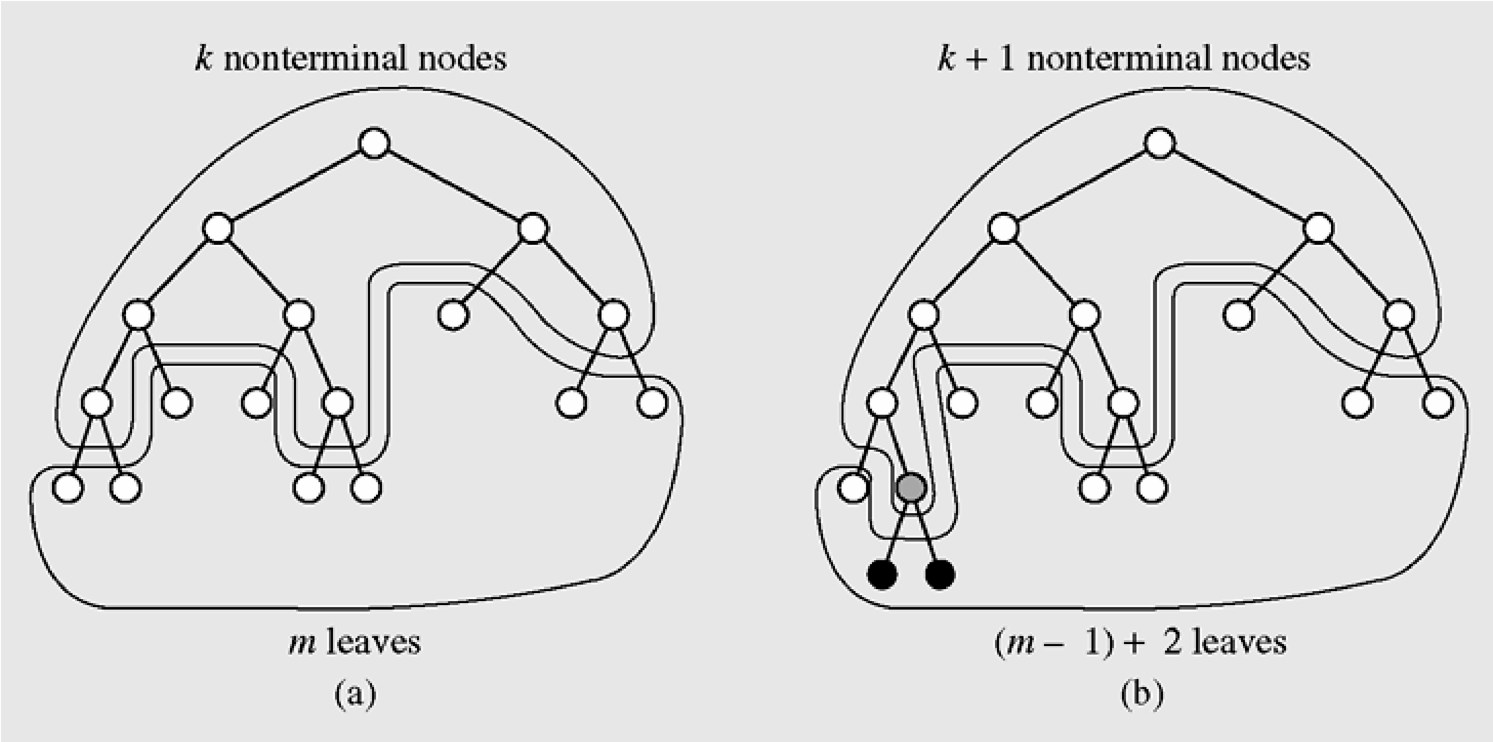


Figure 6-5 Adding a leaf to tree (a), preserving the relation of the number of leaves to the number of nonterminal nodes (b)

ADT Binary Tree (more explicitly)

createBinaryTree() //creates an empty binary tree

createBinary(rootItem) // creates a one-node bin tree whose root contains rootItem

createBinary(rootItem, leftTree, rightTree) //creates a bin tree whose root contains rootItem //and has leftTree and rightTree, respectively, as its left and right subtrees

destroyBinaryTree() // destroys a binary tree

rootData() // returns the data portion of the root of a nonempty binary tree

setRootData(newItem) // replaces the the data portion of the root of a //nonempty bin tree with newItem. If the bin tree is empty, however, //creates a root node whose data portion is newItem and inserts the new //node into the tree

attachLeft(newItem, success) // attaches a left child containing newItem to //the root of a binary tree. Success indicates whether the operation was //successful.

attachRight(newItem, success) // ananlogous to attachLeft

ADT Binary Tree (continued)

attachLeftSubtree(leftTree, success) // Attaches leftTree as the left subtree to the root of a bin tree. Success indicates whether the operation was successful.

attachRightSubtree(rightTree, success) // analogous to attachLeftSubtree

detachLeftSubtree(leftTree, success) // detaches the left subtree of a bin tree's root and retains it in leftTree. Success indicates whether the op was successful.

detachRightSubtree(rightTree, success) // similar to detachLeftSubtree

leftSubtree() // Returns, but does not detach, the left subtree of a bin tree's root

rightSubtree() // analogous to leftSubtree

preorderTraverse(visit) // traverses a binary tree in preorder and calls the function *visit* once for each node

inorderTraverse(visit) // analogous: inorder

postorderTraverse(visit) // analogous: postorder

Implementing Binary Trees

- Binary trees can be implemented in at least two ways:
  - As arrays
  - As linked structures
- To implement a tree as an array, a node is declared as an object with an information field and two “reference” fields

Implementing Binary Trees (continued)

root	free
0	8

Can do for complete binary trees;
A[i] with children A[2i] and A[2i+1].
Parent of A[i] is A[i div 2].

Index	Info	Left	Right
0	13	4	2
1	31	6	–1
2	25	7	1
3	12	–1	–1
4	10	5	3
5	2	–1	–1
6	29	–1	–1
7	20	–1	–1

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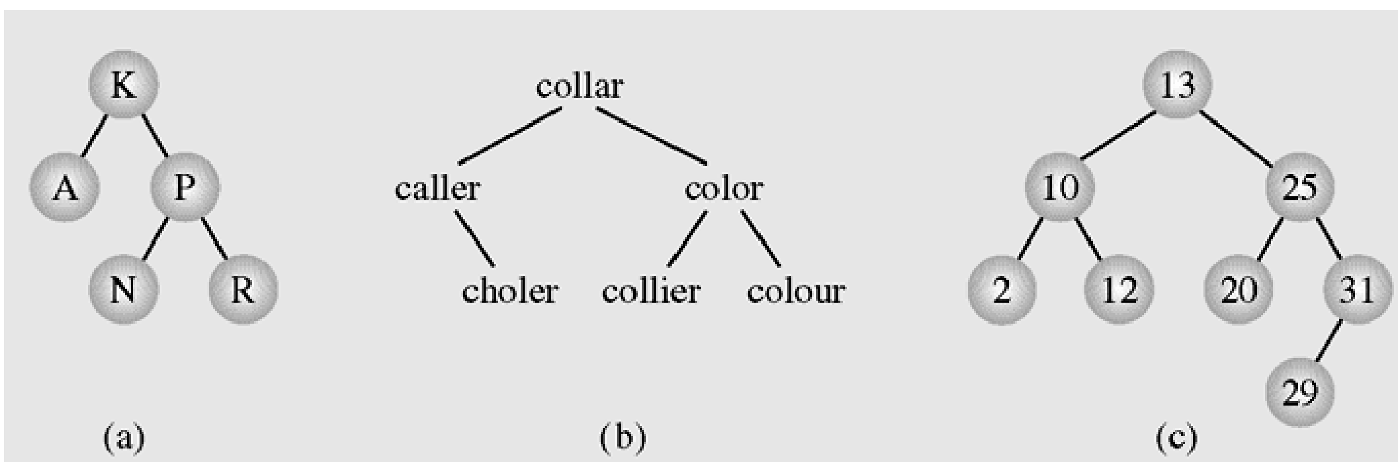
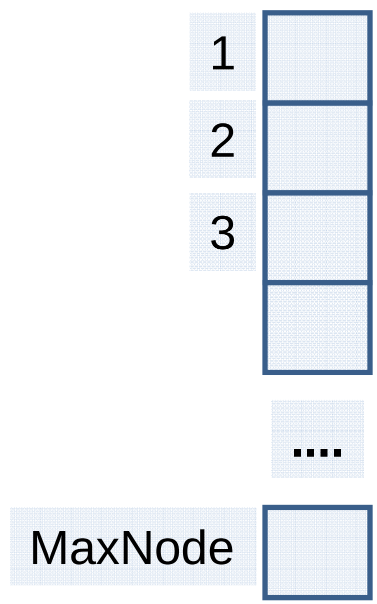


(a) (b) (c)

Implementing Binary Trees (continued)

Can do array for complete binary trees;
Level order storage;
A[i] with children A[2i] and A[2i+1].
Parent of A[i] is A[i div 2]:



Heapsort
Also for trees of max degree k (at most k children)

Binary Tree C++

template

```
template <class T>
struct  TreeNode {
    T         info;
    TreeNode<T> *left, * right;
    int       tag       // a.o. For threading

    TreeNode ( const T& i,
              TreeNode<T> *left = NULL,
              TreeNode<T> *right = NULL )
          : info(i)

          { left = l; right = r; tag = 0; }

};
```

constructor

default

constructor
of type T

See the next slide for the proof of concept; type T=int, is hardwired

The programmed ADT Binary Tree (refers to slide 20, 21: ADT Binary Tree) not parametrized: itemType = int

// ADT

```
#pragma once
#include <iostream>
using namespace std;

struct TreeNode {
    TreeNode * left;
    int data;
    TreeNode * right;
};

class Tree {
public:
    Tree(); // creates empty tree
    Tree(int rootItem);
    Tree(int rootItem, Tree leftTree, Tree rightTree);

    void setRootData(int newItem);

    void attachLeft(int newItem);
    void attachRight(int newItem);

    void attachLeftSubtree(Tree leftTree);
    //void attachRightSubtree(Tree rightTree);

    //void detachLeftSubtree(Tree & leftTree);
    //void detachRightSubtree(Tree & rightTree);

    // more .... see ADT specification

private
    TreeNode * root;

};
```

// Client

```
#include "tree.h"
#include <iostream>
using namespace std;

int main (){
    Tree t;
    t.setRootData(5);
    t.attachLeft(3);
    t.attachRight(7);

    // temporarily made everything public in order to inspect
    cout << "t.root->data: "<< t.root->data  << "\n";
    cout << "t.root -> left -> data: " << t.root->left->data << "\n";
    cout << "t.root -> right -> data: " << t.root->right->data << "\n";
    return 1;
}
```

// Impl.

```
#include "tree.h"

Tree::Tree() {root=0;}

Tree::Tree(int rootItem) {
     TreeNode * root = new TreeNode();
     root -> left =0;
     root -> right=0;
     root -> data=rootItem;
}

Tree::Tree(int rootItem, Tree leftTree, Tree rightTree){
    TreeNode * root = new TreeNode();
    root -> data = rootItem;
    root ->left =0;
    root ->right=0;

    //attachLeftSubtree(leftTree);
    //attachRightSubtree(rightTree);
}
void Tree::setRootData(int newItem) {
    if (root != 0) {
        root -> data = newItem;
    } else {
        root = new TreeNode();
        root -> data = newItem;
        root -> left = 0;
        root ->right =0;
    }
}

void Tree::attachLeft(int newItem){
    if (root != 0){
        if (root ->left == 0) {
            root -> left = new TreeNode();
            root ->left ->data = newItem;
            root ->left->left =0;
            root ->left->right=0;
        }
    }
}
void Tree::attachRight(int newItem){
    if (root != 0){
        if (root ->right == 0) {
            root -> right = new TreeNode();
            root -> right ->data = newItem;
            root -> right->left =0;
            root -> right->right=0;
        }
    }
}
```

Drozdek Does not showHow to implement Generic Binary Tree

FIGURE 6.8 Implementation of a generic binary search tree.

```
//**********************  genBST.h  ************************
//                 generic binary search tree

#include <queue>
#include <stack>

using namespace std;

#ifndef BINARY_SEARCH_TREE
#define BINARY_SEARCH_TREE

template<class T>
class Stack : public stack<T> { ... } // as in Figure 4.21

template<class T>
class Queue : public queue<T> {
public:
    T dequeue() {
        T tmp = front();
        queue<T>::pop();
        return tmp;
    }
    void enqueue(const T& el) {
        push(el);
    }
};

template<class T>
class BSTNode {
public:
    BSTNode() {
        left = right = 0;
    }
    BSTNode(const T& el, BSTNode *l = 0, BSTNode *r = 0) {
        key = el; left = l; right = r;
    }
    T key;
    BSTNode *left, *right;
};
```

FIGURE 6.8 *(continued)*

```
template<class T>
class BST {
public:
    BST() {
        root = 0;
    }
    ~BST() {
        clear();
    }
    void clear() {
        clear(root); root = 0;
    }
    bool isEmpty() const {
        return root == 0;
    }
    void preorder() {
        preorder(root);                     // Figure 6.11
    }
    void inorder() {
        inorder(root);                      // Figure 6.11
    }
    void postorder() {
        postorder(root);                    // Figure 6.11
    }
    T* search(const T& el) const {
        return search(root,el);             // Figure 6.9
    }
    void breadthFirst();                    // Figure 6.10
    void iterativePreorder();               // Figure 6.15
    void iterativeInorder();                // Figure 6.17
    void iterativePostorder();              // Figure 6.16
    void MorrisInorder();                   // Figure 6.20
    void insert(const T&);                  // Figure 6.23
    void deleteByMerging(BSTNode<T>*&);     // Figure 6.29
    void findAndDeleteByMerging(const T&);  // Figure 6.29
    void deleteByCopying(BSTNode<T>*&);     // Figure 6.32
    void balance(T*,int,int);               // Section 6.7
    . . . . . . . . . . . . . . .
protected:
    BSTNode<T>* root;
```

bst

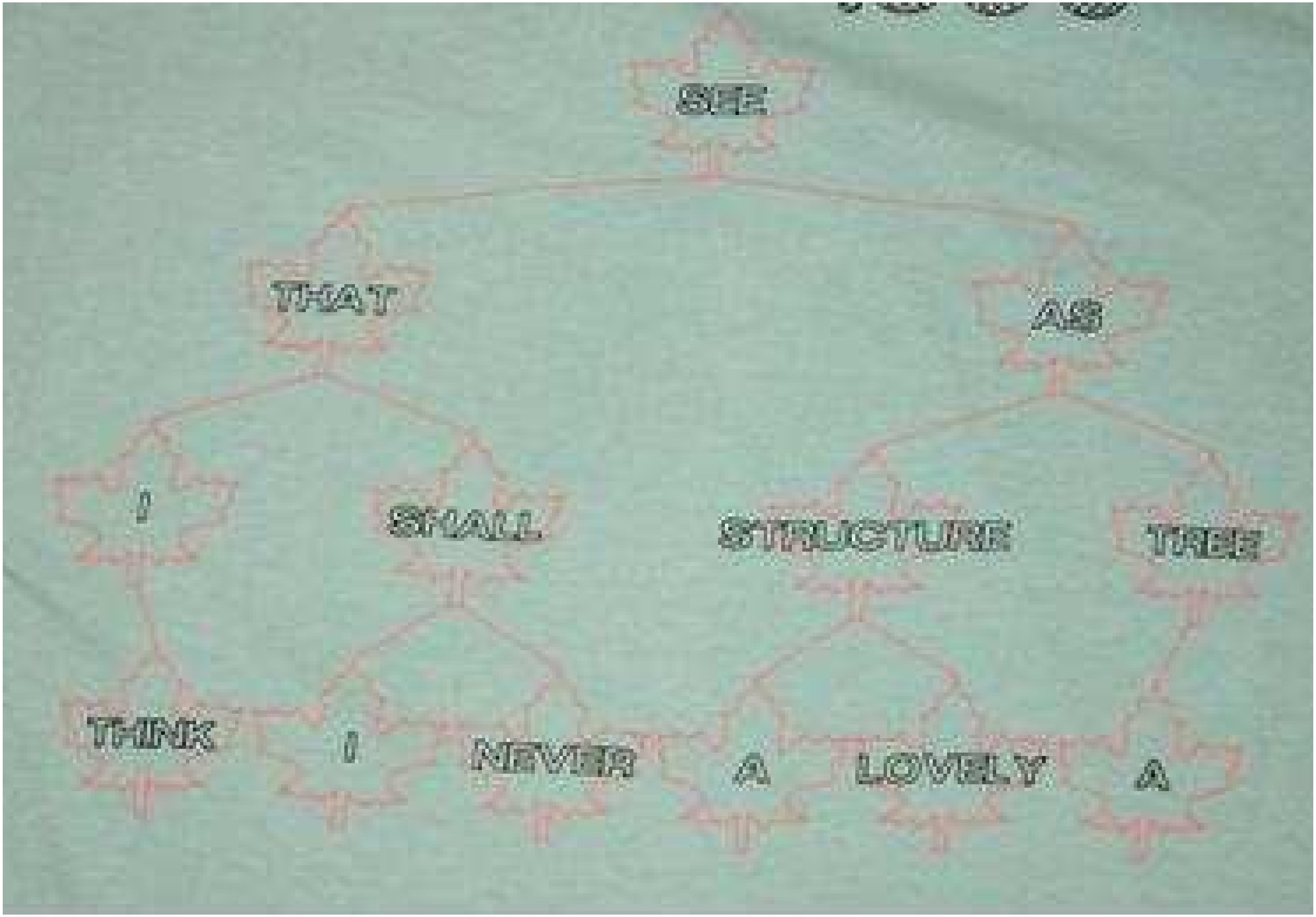
FIGURE 6.8 *(continued)*

bst

```
    void clear(BSTNode<T>*);
    T* search(BSTNode<T>*, const T&) const;  // Figure 6.9
    void preorder(BSTNode<T>*);              // Figure 6.11
    void inorder(BSTNode<T>*);               // Figure 6.11
    void postorder(BSTNode<T>*);             // Figure 6.11
    virtual void visit(BSTNode<T>* p) {
        cout << p->key << ' ';
    }
    . . . . . . . . . . . . . . .
};

#endif
```

Traversal of Binary Trees



Traversal of Binary Trees

Traversals of Binary Trees

- Is the process of visiting each node (precisely once) in a systematic way (visiting has technical meaning, a visit can possibly 'write' to the node, or change the structure of the tree, so you need to do it precisely once for each node; you can 'pass 'by a node many times when are only reading , for instance)
- Why?
  - Get info, updates
  - Check for structural properties, updating
  - Definitely can be extended to graphs (with cycles)!
- Methods:
  - Depth first (recursively or iteratively with stacks):
    - preorder (VLR),
    - inorder(symmetric)-LVR,
    - postorder (LRV)

Traversals of Binary Trees

- Recursively
- Iteratively: stacks (Depth First)
- Queues for Breadth First
- Threaded Trees
- Tree Transformation (e.g., Morris)

Tree Traversal: breadth-first

- **Breadth-first traversal** is visiting each node starting from the lowest (or highest) level and moving down (or up) level by level, visiting nodes on each level from left to right (or from right to left)

Tree Traversal: breadth-first

URE 6.10 Top-down, left-to-right, breadth-first traversal implementation.

```
template<class T>
void BST<T>::breadthFirst() {
    Queue<BSTNode<T>*> queue;
    BSTNode<T> *p = root;
    if (p != 0) {
        queue.enqueue(p);
        while (!queue.empty()) {
            p = queue.dequeue();
            visit(p);
            if (p->left != 0)
                 queue.enqueue(p->left);
            if (p->right != 0)
                 queue.enqueue(p->right);
        }
    }
}
```

Depth-First Traversal

- **Depth-first traversal** proceeds as far as possible to the left (or right), then backs up until the first crossroad, goes one step to the right (or left), and again as far as possible to the left (or right)
  - V — Visiting a node
  - L — Traversing the left subtree
  - R — Traversing the right subtree

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Depth-First Traversal

FIGURE 6.11 Depth-first traversal implementation.

```
template<class T>
void BST<T>::inorder(BSTNode<T> *p) {
     if (p != 0) {
         inorder(p->left);
         visit(p);
         inorder(p->right);
     }
}

template<class T>
void BST<T>::preorder(BSTNode<T> *p) {
    if (p != 0) {
        visit(p);
        preorder(p->left);
        preorder(p->right);
    }
}

template<class T>
void BST<T>::postorder(BSTNode<T>* p) {
    if (p != 0) {
        postorder(p->left);
        postorder(p->right);
        visit(p);
    }
}
```

Inorder Tree Traversal

FIGURE 6.12 Inorder tree traversal.

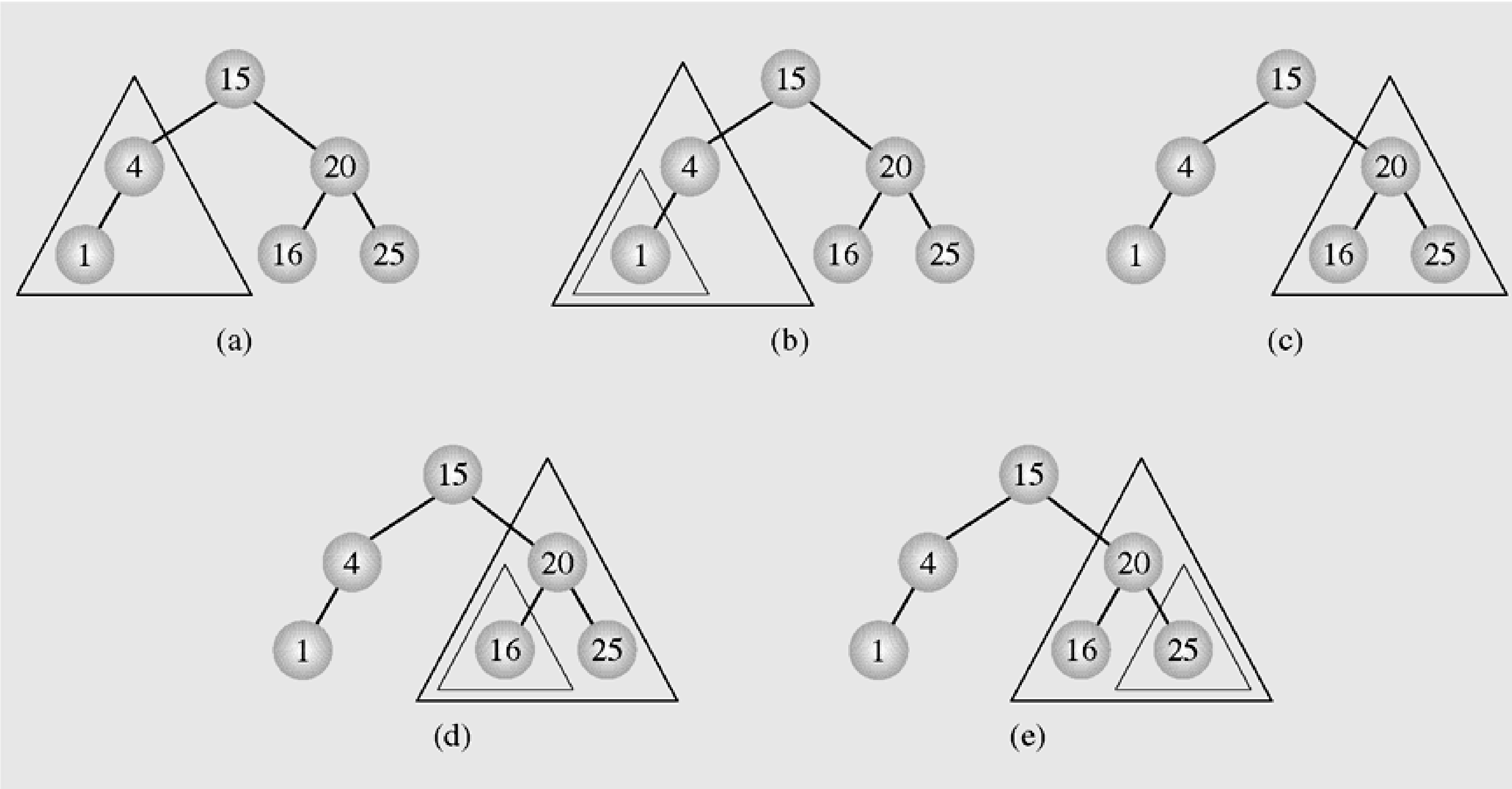


FIGURE 6.17 A nonrecursive implementation of inorder tree traversal.

```
template<class T>
void BST<T>::iterativeInorder() {
    Stack<BSTNode<T>*> travStack;
    BSTNode<T> *p = root;
    while (p != 0) {
        while (p != 0) {           // stack the right child (if any)
            if (p->right)          // and the node itself when going
               travStack.push(p->right); // to the left;
            travStack.push(p);
            p = p->left;
        }
        p = travStack.pop();       // pop a node with no left child
        while (!travStack.empty() && p->right == 0) { // visit it
            visit(p);              // and all nodes with no right
            p = travStack.pop();   // child;
        }
        visit(p);                  // visit also the first node with
        if (!travStack.empty())    // a right child (if any);
             p = travStack.pop();
        else p = 0;
    }
}
```

Preorder Traversal – iterative uses a stack

```
S.create();
S.push(root);
while (not S.isEmpty())  {
   current = S.pop() // a retrieving pop
   while (current ≠ NULL) {
        visit(current);
        S.push(current -> right);
        current = current -> left
   } // end while
} // end while
```

Preorder Traversal – iterative

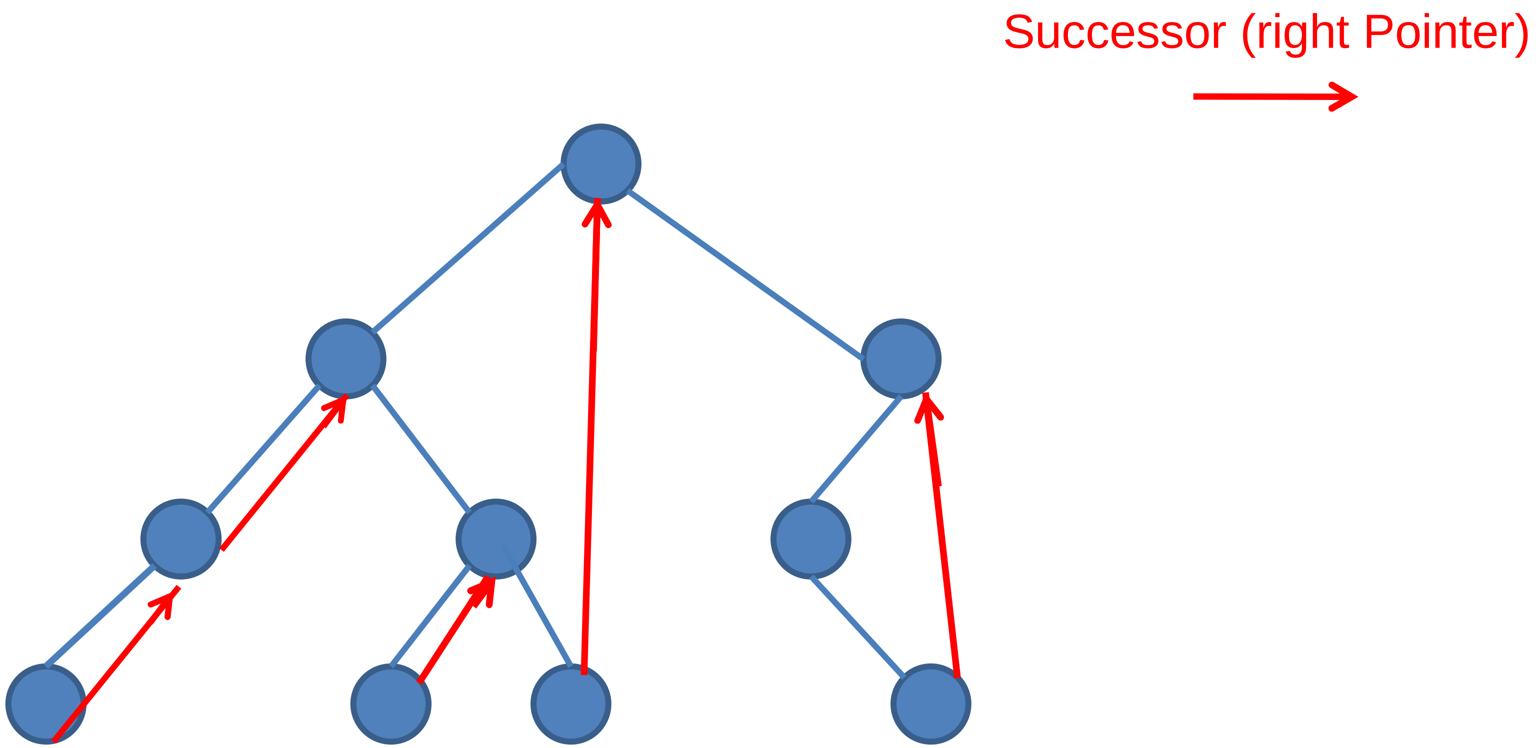
FIGURE 6.15 A nonrecursive implementation of preorder tree traversal.

```
template<class T>
void BST<T>::iterativePreorder() {
    Stack<BSTNode<T>*> travStack;
    BSTNode<T> *p = root;
    if (p != 0) {
        travStack.push(p);
        while (!travStack.empty()) {
            p = travStack.pop();
            visit(p);
            if (p->right != 0)
                 travStack.push(p->right);
            if (p->left != 0)   // left child pushed after right
                 travStack.push(p->left); // to be on the top of
        }                           // the stack;
    }
}
```

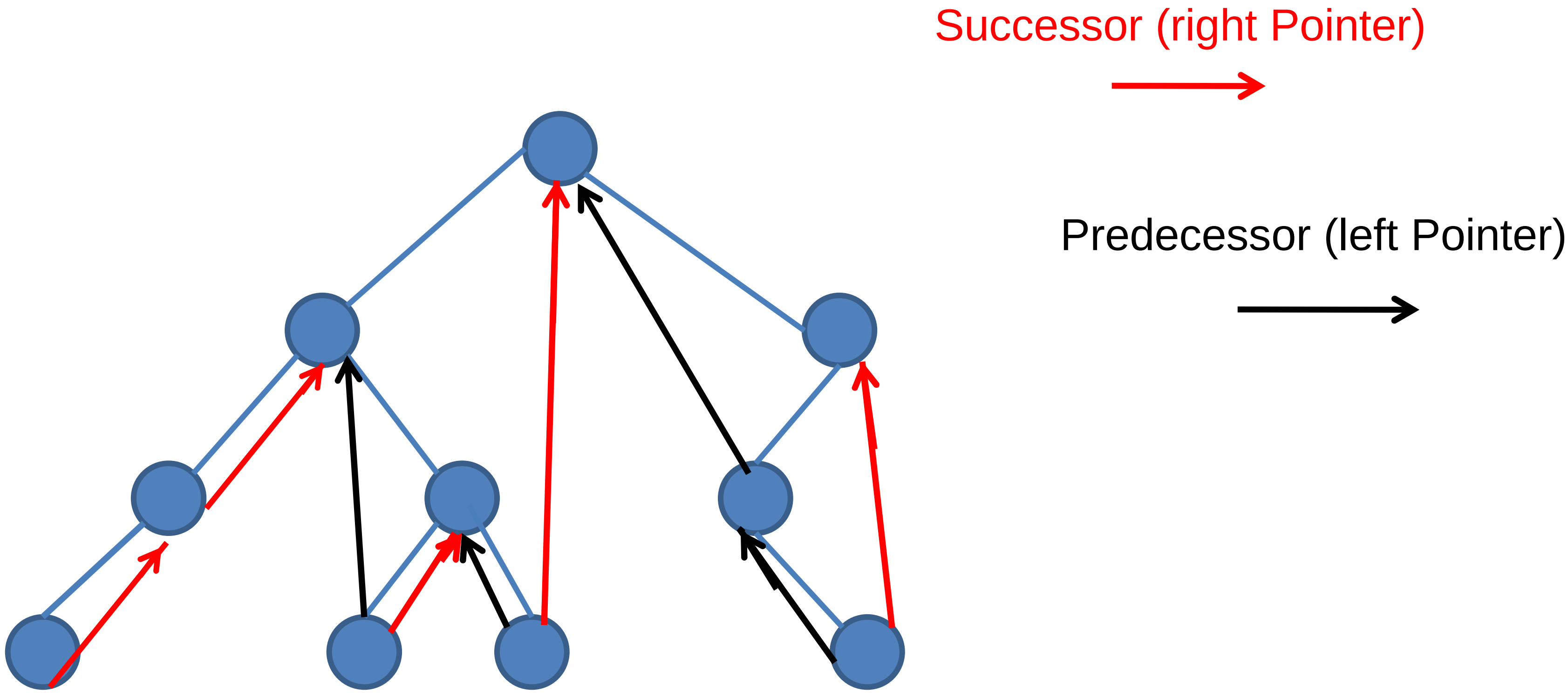
Stackless Depth-First Traversal

- **Threads** are references to the predecessor and successor of the node according to an inorder traversal
- Trees whose nodes use threads are called **threaded trees**

A threaded tree; an inorder traversal´s path in a threaded tree with Right successors only



A threaded tree; right pointers: successors; left pointers: predecessors



Traversal Through Tree Transformation

MorrisInOrder ()

while *not finished*

    if *node has **NO** left descendant*

        visit it;

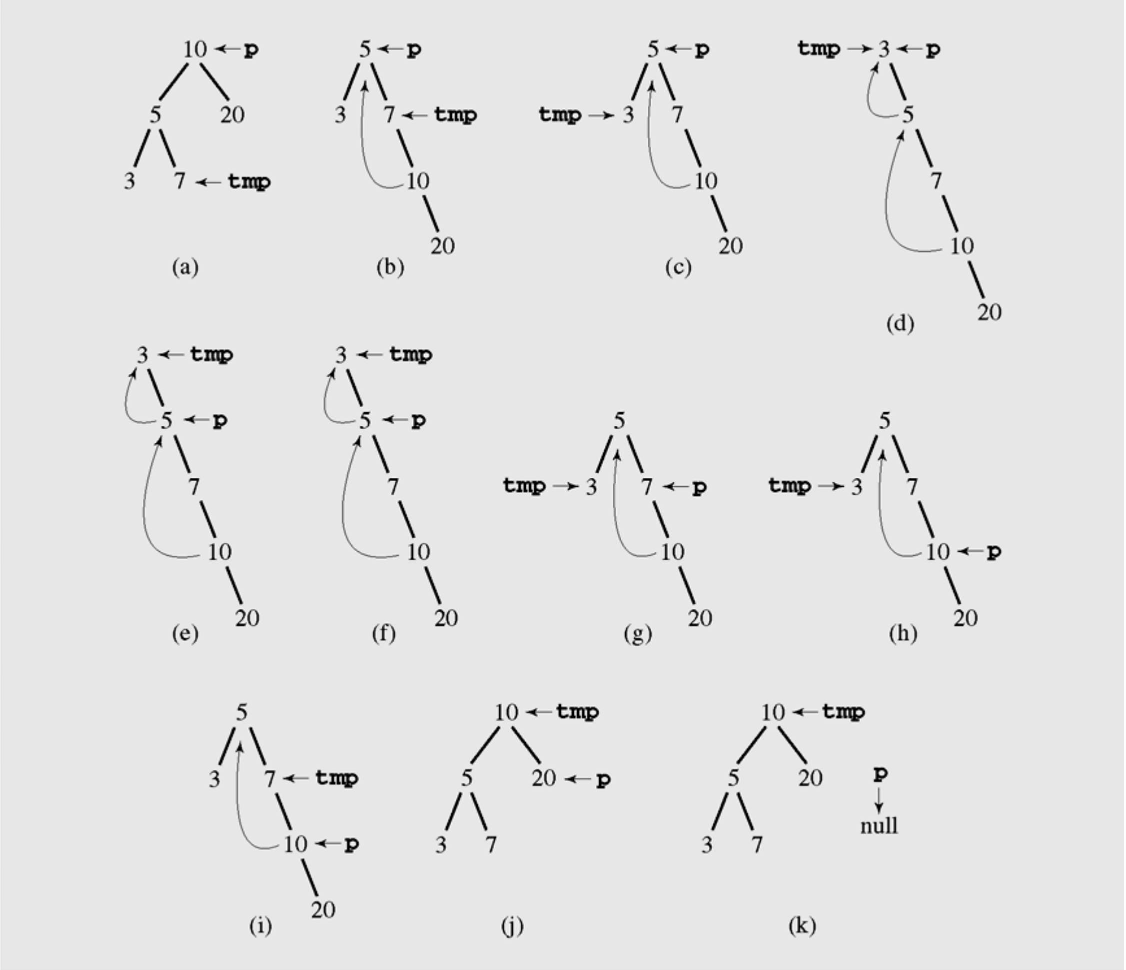
        go to the right;

    else *make this node the right child of the rightmost node*

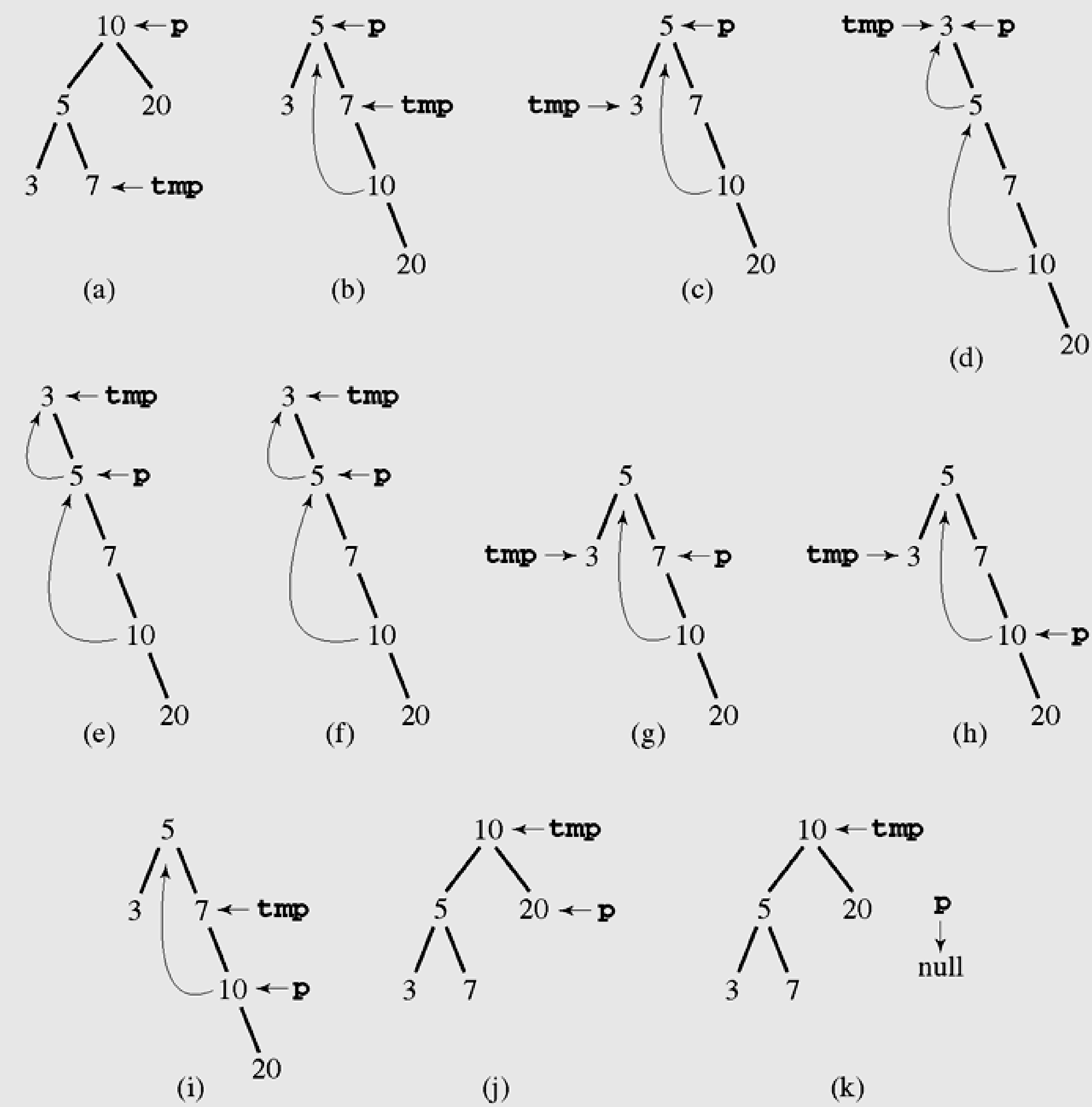
            in its left descendant;

            go to this left descendant

FIGURE 6.21 Tree traversal with the Morris method.



Traversal Through Tree Transformation



Traversal Through Tree Transformation

FIGURE 6.20 Implementation of the Morris algorithm for inorder traversal.

```
template<class T>
void BST<T>::MorrisInorder() {
    BSTNode<T> *p = root, *tmp;
    while (p != 0)
        if (p->left == 0) {
             visit(p);
             p = p->right;
        }
        else {
             tmp = p->left;
             while (tmp->right != 0 && // go to the rightmost node
                       tmp->right != p)   // of the left subtree or
                    tmp = tmp->right;     // to the temporary parent
             if (tmp->right == 0) {       // of p; if 'true'
                    tmp->right = p;       // rightmost node was
                    p = p->left;          // reached, make it a
             }                            // temporary parent of the
             else {                       // current root, else
                                          // a temporary parent has
                    visit(p);             // been found; visit node p
                    tmp->right = 0;       // and then cut the right
                                          // pointer of the current
                    p = p->right;         // parent, whereby it
             }                            // ceases to be a parent;
        }
}
```

Figure 6-20 Implementation of the Morris algorithm for inorder traversal

Binary Search Trees

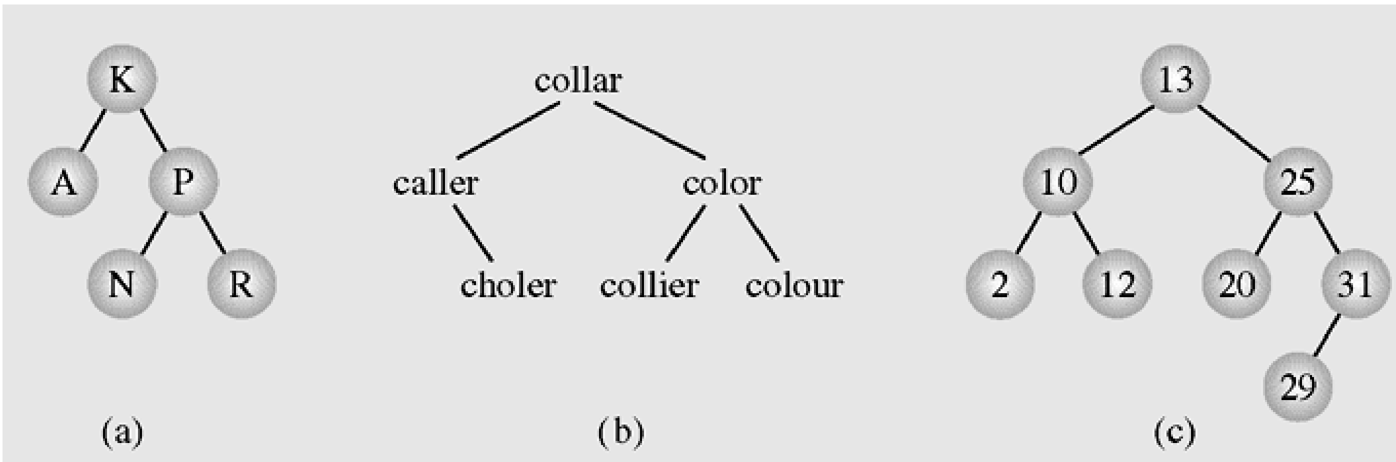


Figure 6-6 Examples of binary search trees

Searching a Binary Search Tree (continued)

- The **internal path length (IPL)** is the sum of all path lengths of all nodes
- It is calculated by summing $\Sigma(i - 1)l_i$ over all levels *i,* where l_i is the number of nodes on level *l*
- A depth of a node in the tree is determined by the path length
- An average depth, called an **average path length***,* is given by the formula IPL/*n,* which depends on the shape of the tree

Insertion

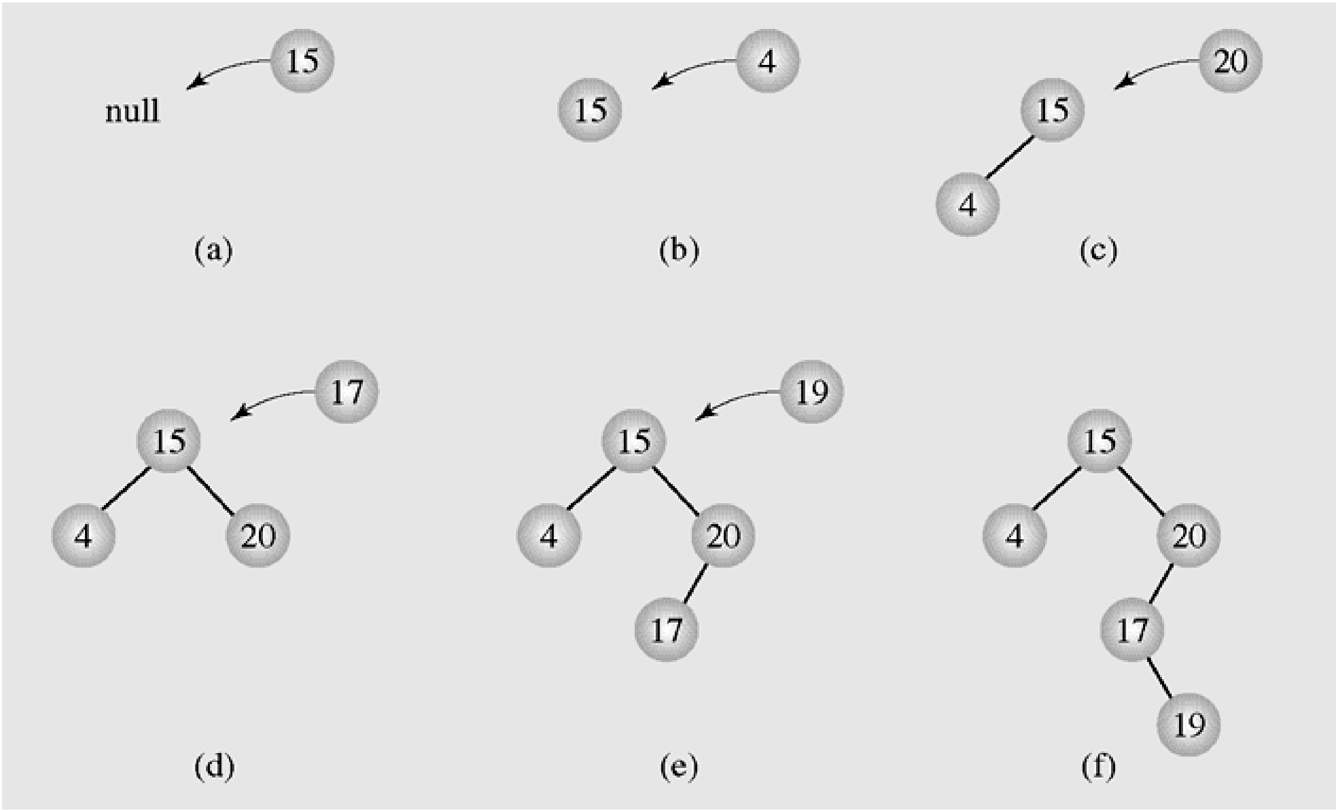


Figure 6-22 Inserting nodes into binary search trees

Insertion (continued)

FIGURE 6.23 Implementation of the insertion algorithm.

```
template<class T>
void BST<T>::insert(const T& el) {
    BSTNode<T> *p = root, *prev = 0;
    while (p != 0) {         // find a place for inserting new node;
        prev = p;
        if (p->key < el)
             p = p->right;
        else p = p->left;
    }
    if (root == 0)    // tree is empty;
         root = new BSTNode<T>(el);
    else if (prev->key < el)
         prev->right = new BSTNode<T>(el);
    else prev->left  = new BSTNode<T>(el);
}
```

Figure 6-23 Implementation of the insertion algorithm

Insertion (continued)

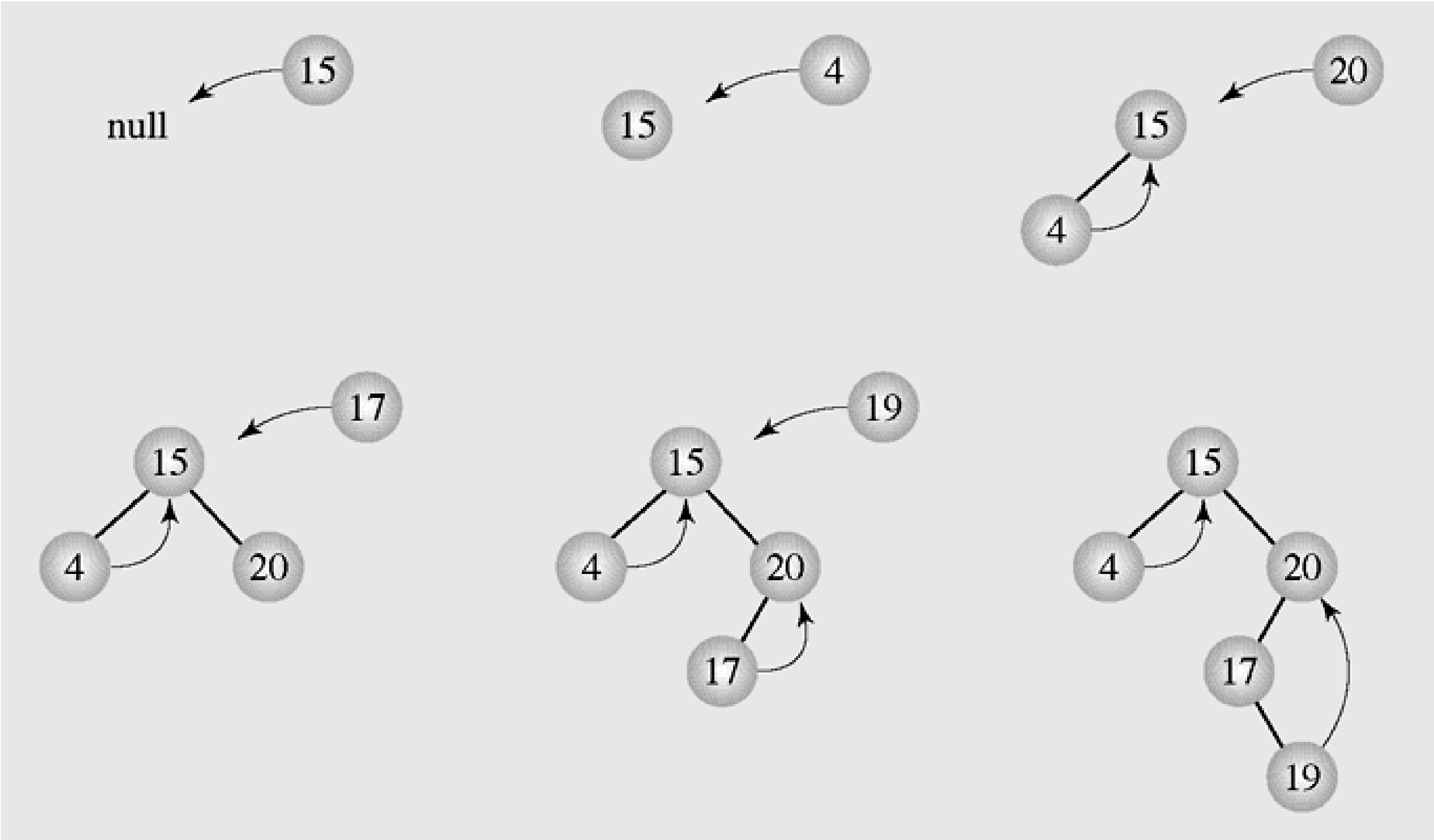


Figure 6-25 Inserting nodes into a threaded tree

Deletion in BSTs

- There are three cases of deleting a node from the binary search tree:
  - The node is a leaf; it has no children
  - The node has one child
  - The node has two children

Deletion (continued)

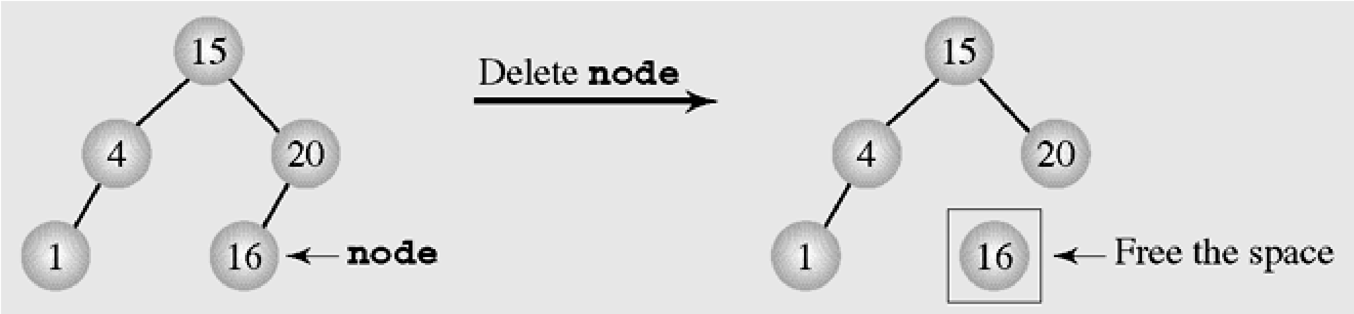


Figure 6-26 Deleting a leaf

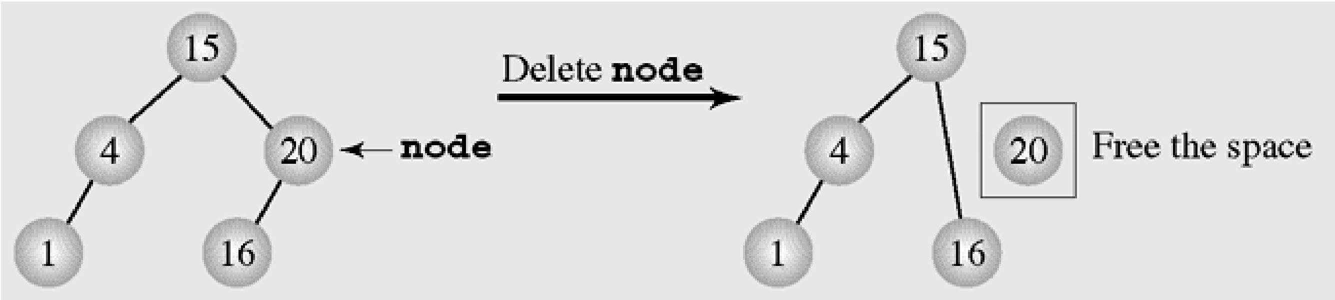


Figure 6-27 Deleting a node with one child

Deletion by Merging

- Making one tree out of the two subtrees of the node and then attaching it to the node's parent is called **deleting by merging**

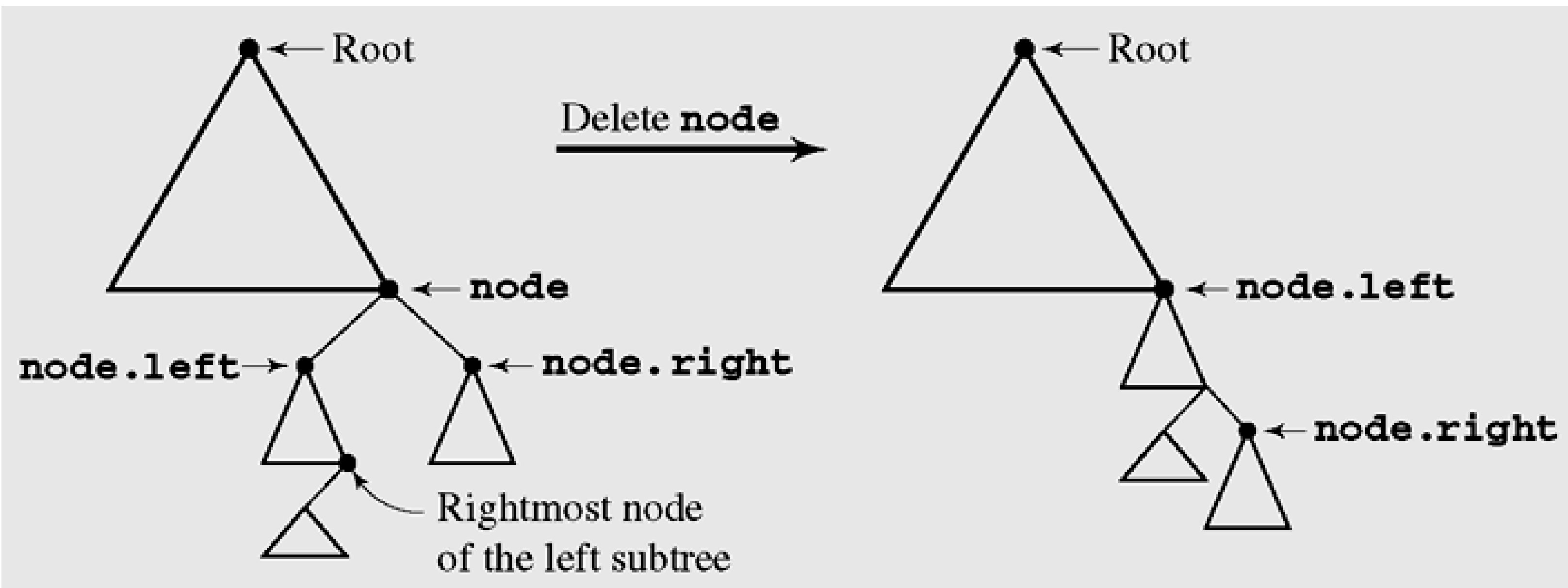


Figure 6-28 Summary of deleting by merging

Deletion by Copying

- If the node has two children, the problem can be reduced to:
  - The node is a leaf
  - The node has only one nonempty child
- Solution: replace the key being deleted with its immediate predecessor (or successor)
- A key's predecessor is the key in the rightmost node in the left subtree